Smooth Rationalization

Online Appendix

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This appendix presents smooth rationalization results focusing on utility functions with further structure than being strictly increasing and concave. We study three specific cases: strictly concave utilities (Section I), homothetic utilities (Section II), and quasilinear utilities (Section III).[1] In all cases, we can achieve smooth rationalization by the same approach as in the general case; the only additional condition for smooth rationalization is to smooth the indifference sets. The main difference is that different structures on the utility functions allow us to infer different indifferences. All proofs are in Section IV.

I Smooth Rationalization by a Strictly Concave Utility

In this section, we analyze conditions to smoothly rationalize the observed choices by a strictly concave utility, i.e., for smooth rationalization restricted to SARP. A strictly concave utility implies that each chosen bundle must be the *unique* optimal choice from its budget set, i.e., the demand is single-valued.[2] The implication for

[1] Homothetic and quasilinear utilities are the only cases in which the consumer surplus is a valid welfare measure (Silberberg, 1990, see Section 11.5 in).

[2] To see this, suppose x^i is an optimal choice from price p^i, but there is another bundle x that is affordable ($p^i \cdot x \leq 1$) and indifferent to x^i. Take $z = {}^{(x+x^i)}/_2$. Then $p^i \cdot z \leq 1$ and, as U is strictly concave $U(z) > U(x^i)$, which contradicts the optimality of x^i.

revealed preferences is that different chosen bundles cannot revealed indifferent to each other.[3]

The main implication of a strictly concave utility for our data set modification is that one application of the modification is enough to reach the fixed point.

Lemma A.1. *If $\Gamma(\mathcal{D})$ is rationalizable by a strictly concave utility, then $\Gamma(\mathcal{D}) = \Gamma(\Gamma(\mathcal{D}))$.*

The following result is an equivalent of Proposition 2 for the case of a strictly concave utility.

Proposition A.1. *Suppose $\mathcal{D} = \Gamma(\mathcal{D})$. Then $\mathcal{D}$ is rationalizable by a strictly concave utility if, and only if, it is smoothly rationalizable by a strictly concave and infinitely differentiable utility.*

Theorem A.1 presents the characterization of smooth rationalization by a strictly concave utility.

Theorem A.1. *The following are equivalent:*

Sa-1) *$\mathcal{D}$ is smoothly rationalizable by a strictly concave utility.*

Sa-2) *$\Gamma(\mathcal{D})$ is rationalizable by a well-behaved and strictly concave utility.*

Sa-3) *For all $i \in [N]$ there exists numbers $u^i \in \mathbb{R}$, $\lambda^i > 0$ and K-dimensional vectors $\mu^i \geq \mathbf{0}$ such that*

$$u^i > u^j + \lambda^i(1 - p^i \cdot x^j) + \mu^i \cdot x^j \qquad \text{whenever } x^i \neq x^j \tag{S1}$$

$$u^i = u^j \qquad \text{whenever } x^i = x^j \tag{S2}$$

$$\lambda^i p^i - \mu^i = \lambda^j p^j - \mu^j \qquad \text{whenever } x^i = x^j \tag{S3}$$

$$\lambda^i p^i - \mu^i \gg \mathbf{0} \qquad \text{for all } i \in [N] \tag{S4}$$

$$\mu^i \cdot x^i = 0 \qquad \text{for all } i \in [N] \tag{S5}$$

[3] To see this, note that rationalization by a strictly concave utility is equivalent to SARP, which rules out revealed indifferences between different chosen bundles.

Sa-4) $\mathcal{D}$ *is smoothly rationalizable by a strictly concave and infinitely differentiable utility.*

The interpretation of the statements in Theorem A.1 is similar to the ones in Theorem 1. One particular difference with rationalization results that do not require smoothness is that both (S1) and (G1) is that both are strict inequalities. In contrast, the Afriat inequalities are weak in the case of a concave utility (see Afriat, 1967; Varian, 1982), and strict in the case of a strictly concave utility (Theorem 2 in Matzkin & Richter, 1991). The reason for this difference is that, as explained in Section 3, smooth rationalization precludes us from introducing further indifferences than the ones implied by the revealed preferences, which is also true when we require rationalization by a well-behaved and strictly concave utility even when smoothness is not required.

An important comment regarding testing the smooth rationalization by a strictly concave utility is that, in the modified data set $\Gamma(\mathcal{D})$, SARP and rationalization by a well-behaved and strictly concave utility are not equivalent. The reason for this is that, on the one hand, SARP does not compare a bundle with itself and, on the other, $\Gamma(\mathcal{D})$ might have some observation i satisfying $r^i \cdot x^i < 1$, violating strict monotonicity. An example of this problem is presented in Figure A.1, which presents the modified data set of the panel (a) of Figure 1. This modified data set vacuously satisfies SARP; however, it is not rationalizable by any strictly increasing utility as the chosen bundle is in the interior of the (modified) budget set.

The following proposition presents a simple test for rationalizing the modified data set by a strictly concave utility. This result is a direct consequence of Theorem 2 in Matzkin and Richter (1991); hence we omit the proof.

Proposition A.2. $\Gamma(\mathcal{D})$ *is rationalizable by a well-behaved and strictly concave utility if, and only if, it satisfies SARP and* $r^i \cdot x^i = 1$ *for all* $i \in [N]$.

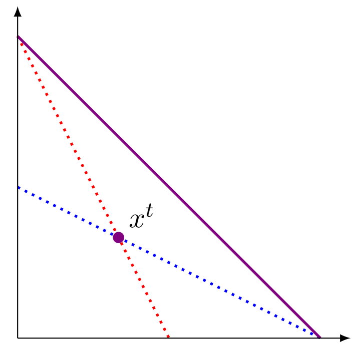

Figure A.1: Modified data set of panel (a) in Figure 1.

II Smooth Rationalization by a Homothetic Utility

Homothetic utilities have the property that the marginal rate of substitutions depends only on the share between goods, not their level. Hence, the ratio of goods in the optimal consumption bundle depends only on the price ratio, not the income level.

The rationalization test for homothetic utilities has a cyclical structure. To achieve smooth rationalization, we first show that the cycles in the test can be used to infer indifferences in the data beyond the ones obtained by the revealed indifference relation. After that, we use all the inferred indifferences to modify the data set by taking the meet of prices among indifferences (until finding a fixed point). As in the previous sections, the test for differentiability of the utility function is the original rationalization test applied to the modified data set.

A utility function u is *homothetic* if it is a monotonic transformation of a function that is homogeneous of degree one, i.e., if $u(x) = f(g(x))$, where $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and $g : \mathbb{R}^K_+ \to \mathbb{R}$ is homogeneous of degree one. Since utility representations are invariant to monotonic transformations (utility is an ordinal representation of preferences), we focus on utilities that are homogeneous of degree one. Hence, we use the terms homothetic and homogeneous of degree one interchangeably. One caveat of assuming homotheticity is that we cannot obtain differentiability at the zero bundle; this is because any function that is homogeneous of degree one and differentiable at $\mathbf{0}$ is linear, which is a stronger requirement.[4] Hence, in this section, we say that a data set is smoothly rationalizable by a homothetic utility if such utility is well-behaved, homothetic, and differentiable in $\mathbb{R}^K_+ \setminus \mathbf{0}$.

Varian (1983) shows that the Homothetic Axiom of Revealed Preferences, HARP, is a test for rationalization by a homothetic utility.[5]

Definition A.1. $\mathcal{D}$ satisfies the *Homothetic Axiom of Revealed Preferences, HARP*, if for any sequence of different observations $(m_\ell)_{\ell \in [L]}$,

$$(p^{m_L} \cdot x^{m_1})(p^{m_1} \cdot x^{m_2})(p^{m_2} \cdot x^{m_3}) \dots (p^{m_{L-1}} \cdot x^{m_L}) \geq 1\,.$$

Furthermore, Varian (1983) shows that if $\mathcal{D}$ is rationalizable by a homothetic utility, such utility can always be chosen to be well-behaved and that there are numbers $u^i > 0$ such that $u^i \leq u^j\, p^j \cdot x^i$ for all i, j.[6] Knoblauch (1993) characterizes all homothetic preferences that rationalize $\mathcal{D}$ for a given data set that satisfies HARP.

We present an example to understand the motivation for HARP and the restrictions it imposes to achieve differentiability. Suppose we have three observations i, j, m such that

$$(p^i \cdot x^j)(p^j \cdot x^m)(p^m \cdot x^i) = 1$$

and a homothetic utility U rationalizing such choices. As $p^i \cdot \big((p^j \cdot x^m)(p^m \cdot x^i) x^j\big) =$

[4]To see that a function that is homogeneous of degree one and differentiable at $\mathbf{0}$ is linear, take f to satisfy both properties. Since for $\lambda > 0$ we have $f(\mathbf{0}) = f(\lambda \mathbf{0}) = \lambda f(\mathbf{0})$ we conclude $f(\mathbf{0}) = 0$. Fix $x \neq \mathbf{0}$; since f is differentiable at zero

$$0 = \lim_{t \to 0^+} \frac{f(t\,x) - f(\mathbf{0}) - \nabla f(\mathbf{0}) \cdot (t\,x - \mathbf{0})}{||t\,x - \mathbf{0}||} = \lim_{t \to 0^+} \frac{t\,f(x) - t\,\nabla f(\mathbf{0}) \cdot x}{t\,||x||} = \frac{f(x) - \nabla f(\mathbf{0}) \cdot x}{||x||}\,.$$

Hence $f(x) = \nabla f(\mathbf{0}) \cdot x$. Since x is arbitrary, we conclude that f is linear.

[5]When characterizing homothetic utility, Varian (1983) points to several bibliographical remarks regarding previous related results, in particular, he states that Afriat (1981) already presented a version of rationalization by a homothetic utility.

[6]The motivation for these numbers is that if U is homothetic, then the marginal utility of income λ equals the utility $U(x)$. The numbers can be obtained by replacing this equality on the Afriat inequalities.

1, revealed preferences imply $U(x^i) \geq U\left((p^j \cdot x^m)(p^m \cdot x^i)x^j\right)$. Moreover, since U is homothetic and x^j is the optimal choice from p^j, then $(p^j \cdot x^m)(p^m \cdot x^i)x^j$ is optimal from $\left[(p^j \cdot x^m)(p^m \cdot x^i)\right]^{-1} p^j$; in particular, as $\left[(p^j \cdot x^m)(p^m \cdot x^i)\right]^{-1} p^j \cdot (p^m \cdot x^i)x^m = 1$, we have $U\left((p^j \cdot x^m)(p^m \cdot x^i)x^j\right) \geq U\left((p^m \cdot x^i)x^m\right)$. Similarly, as x^m is optimal from p^m, homotheticity implies that $(p^m \cdot x^i)x^m$ is optimal from $(p^m \cdot x^i)^{-1} p^m$, thus $U\left((p^m \cdot x^i)x^m\right) \geq U(x^i)$. Collecting these inequalities yields

$$U(x^i) \geq U\left((p^j \cdot x^m)(p^m \cdot x^i)x^j\right) \geq U\left((p^m \cdot x^i)x^m\right) \geq U(x^i) \tag{1}$$

If choices are rationalizable by a homothetic utility, then the previous (weak) inequalities must be equalities. Thus, if instead of starting from the product being equal to one, we start from $(p^i \cdot x^j)(p^j \cdot x^m)(p^m \cdot x^i) < 1$, then the first inequality in the above restriction would be strict, implying that rationalizing $\mathcal{D}$ with a homothetic utility is not possible.

Since rationalization by a homothetic utility implies that the inequalities in (1) have to be equalities, we can infer indifferences between x^i and projections of x^j and x^m. Furthermore, if we assume differentiability, we obtain that x^i, x^j, and x^m have the same MRS (for interior solutions). Since $(p^j \cdot x^m)(p^m \cdot x^i)x^j$ is affordable at price p^i, then it is optimal from such price. Hence, the MRS at this bundle (and thus also at x^j) is equal to the price ratio in p^i, which is also equal to the MRS at x^i. Similarly, we can conclude equal MRS at x^i and $(p^m \cdot x^i)x^m$, hence also at x^m. Figure A.2 presents a simple example in which a homothetic rationalization exists but cannot be smooth.

As in the previous sections, the critical component to characterize rationalization by a differentiable utility is the existence of indifferences. For the reasons explained in the previous paragraph, we can infer indifferences by looking at sequences of observations like the ones analyzed in the definition of HARP. Also, the indifferences inferred from the revealed indifference relation in Definition 1 can also be

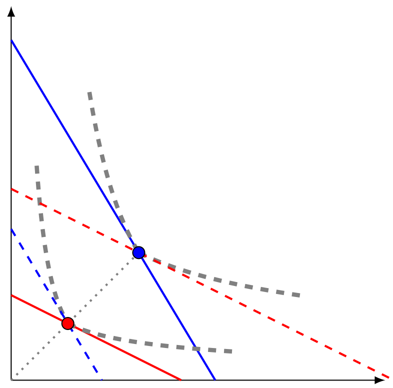

Figure A.2: The data set can be rationalized by a homothetic utility. However, it cannot be differentiable as both choices cannot have the same MRS.

inferred through the cycles analyzed in HARP.[7] Hence, it is sufficient to only focus on such cycles. We will refer to sequences that allow us to infer indifferences in the homothetic case as H1-sequences (i.e., HARP sequences that are less or equal to one).

Definition A.2. An *H1-sequence* is a sequence of unique observations $(m_\ell)_{\ell \in [L]}$ such that

$$(p^{m_L} \cdot x^{m_1})(p^{m_1} \cdot x^{m_2}) \dots (p^{m_{L-1}} \cdot x^{m_L}) \leq 1$$

The reason to include a weak inequality in the previous definition (instead of equality) is analogous to the idea of not changing the definition of revealed indifferences in the modification of the data set (Definition 7). Although the inequality in the previous definition has to be an equality to satisfy HARP, defining the sequences in this form will ensure that creating the modified data set for homothetic rationalization stops in a finite number of steps. Furthermore, suppose that we have an H1-sequence for which the inequality is strict after the data set modification. In that case, we already know that a homothetic utility cannot smoothly rationalize such a data set.

[7]Suppose $(p^i \cdot x^j)(p^j \cdot x^m)(p^m \cdot x^i) = 1$ and we know $x^i \sim x^j \sim x^m$. GARP implies (without loss of generality) $p^i \cdot x^j = p^j \cdot x^m = p^m \cdot x^i = 1$. Therefore $U(x^i) = U((p^j \cdot x^m)(p^m \cdot x^i)x^j) = U((p^m \cdot x^i)x^m)$ is equivalent to $U(x^i) = U(x^j) = U(x^m)$.

The motivation for creating a modified data set is that indifferences between choices in the data imply equality of the MRSs. Along with concavity, this equality makes the indifference set linear. Since assuming a homothetic utility implies that we can infer indifferences between one choice and a scaled version of another (through H1-sequences), we must add such indifferences in our analysis. To simplify our notation, we will focus directly on prices instead of choices. Following Varian (1982) (Section 3), we refer to a price as revealed indifferent to another if we can infer that the optimal choice from the former is indifferent to the optimal choice of the latter.

Suppose again $(p^i \cdot x^j)(p^j \cdot x^m)(p^m \cdot x^i) = 1$. Focusing on observation i, this example generates the H1-sequence $s = (j, k, i)$. Since we infer that $(p^j \cdot x^m)(p^m \cdot x^i)x^j$ is both optimal from $(p^j \cdot x^m)(p^m \cdot x^i)p^j$ and indifferent to x^i, the agent is indifferent between prices p^i and $(p^j \cdot x^m)(p^m \cdot x^i)x^j$. Similarly, she is also indifferent between p^i and $(p^m \cdot x^i)p^m$. These indifferences imply that all these prices should be considered when modifying the price of observation i. The following definition formalizes such a notion.

Definition A.3. For any H1-sequence $s = (m_\ell)_{\ell \in [L]}$ let

$$I^H(s) = \left\{ \left(\prod_{r=\ell}^{L-1} p^{m_r} \cdot x^{m_{r+1}} \right)^{-1} p^{m_\ell} \right\}_{\ell \in [L]}$$

be the set of prices that are revealed indifferent to p^{m_L}. Furthermore, denote by $\mathcal{S}^H(i)$ the set of all H1-sequences ending in i. The *homothetic modification* of $\mathcal{D}$, $\Gamma^H(\mathcal{D})$, is $\Gamma^H(\mathcal{D}) = \left(r^{i,H}, x^i\right)_{i \in [N]}$, where

$$r^{i,H} = \bigwedge \bigcup_{s \in \mathcal{S}^H(i)} I^H(s)\,.$$

Every H1-sequence s ending in i yields information about prices that are indifferent to p^i; the previous definition collects all such prices in $I^H(s)$. We take all the prices revealed indifferent to p^i by taking the union of all the sets $I^H(s)$ for sequences that end in i. Finally, the homothetic modification of $\mathcal{D}$, $\Gamma^H(\mathcal{D})$, replaces each price by the meet of all prices revealed indifferent to p^i.

The following result, analogous to Proposition 1, shows the role of the fixed point of the homothetic modification to test for a differentiable utility.

Proposition A.3. *Suppose $\mathcal{D} = \Gamma^H(\mathcal{D})$. $\mathcal{D}$ is rationalizable by a well-behaved and homothetic utility if, and only it, it is smoothly rationalizable by a homothetic utility that is infinitely differentiable everywhere except at $\mathbf{0}$.*

As the proofs of Proposition 2 and Proposition A.1, the proof of the previous result is constructive. However, to construct a homothetic rationalizing utility requires a further step. The reason is that the convolution technique used in the proofs does not preserve homotheticity. For this construction, we start by using the function proposed by Varian (1983) (using the numbers in H-3) instead of the ones he proposed), which is homothetic. Then, we project all the choices into one indifference curve on this function and use convolution to smooth that particular indifference curve. Then, we expand that smooth indifference curve to other utility levels. When doing it, we choose a superset of $\mathbb{R}^K_+$ that assures such an extension is well-defined. This expansion yields a (well-behaved) homothetic utility function. Finally, following Debreu (1972) and Neilson (1991), we conclude that this utility is also infinitely differentiable everywhere except at $\mathbf{0}$.

Theorem A.2 presents our characterization of rationalization by a homothetic and differentiable utility.

Theorem A.2. *Let $\mathcal{D}^H_\wedge$ be the fixed point of Γ^H, starting from $\mathcal{D}$. The following are equivalent:*

H-1) $\mathcal{D}$ is smoothly rationalizable by a homothetic utility.

H-2) $\mathcal{D}^H_\wedge$ is rationalizable by a homothetic utility (i.e., satisfies HARP).

H-3) For two bundles x^i, x^j, denote by $x^i \approx_\wedge^H$ the existence of an H1-sequence in $\mathcal{D}_\wedge^H$ containing both i and j. There are numbers $u^i > 0$ and $\mu^i \geq \mathbf{0}$ such that

$$u^i p^i \cdot x^j > u^j + \mu^i \cdot x^j \qquad \textit{whenever } x^i \not\approx^H x^j \tag{H1}$$

$$u^i p^i \cdot x^j = u^j + \mu^i \cdot x^j \qquad \textit{whenever } x^i \approx^H x^j \tag{H2}$$

$$u^i p^i - \mu^i = u^j p^j - \mu^j \qquad \textit{whenever } x^i \approx^H x^j \tag{H3}$$

$$u^i p^i - \mu^i \gg \mathbf{0} \qquad \textit{for all } i \in [N] \tag{H4}$$

$$\mu^i \cdot x^i = 0 \qquad \textit{for all } i \in [N] \tag{H5}$$

H-4) $\mathcal{D}$ is smoothly rationalizable by a homothetic utility that is infinitely differentiable everywhere except at $\mathbf{0}$.

III Smooth Rationalization by a Quasilinear Utility

A utility function is quasilinear if it takes the form $U(x) + y$, where $x \in \mathbb{R}_+^K$ is the commodities space, and y is a numeraire good. The numeraire good is measured in the same units as wealth. Hence its price is one. Quasilinear utility functions are used in many areas of economics, including mechanism design, public economics, industrial organization, and international trade. In all such applications, quasilinear utilities are usually also assumed to be differential. We can think of an agent as maximizing a quasilinear utility if the following terms:

Definition A.4. $\mathcal{D}$ is *quasilinear rationalizable* if there is a well-behaved utility U such that for all $i \in [N]$, x^i maximizes $U(x) + y$ subject to $p^i \cdot x + y = 0$ and $x \geq \mathbf{0}$. The function $U(x) + y$ *quasilinear rationalizes* $\mathcal{D}$

As in many applications, we assume that the numeraire good can be negative; hence, the budget constraint is nonbinding in the maximization problem (see Chapter 4.2.3 in Chambers & Echenique, 2016). The previous definition implies that $\mathcal{D}$ is rationalizable by a quasilinear utility U if, and only if, $U(x^i) - 1 \geq U(x) - p^i \cdot x$

for all $i \in [N]$ and $x \in \mathbb{R}^K_+$. Brown and Calsamiglia (2007) show that the rationalization by a quasilinear utility is equivalent to cyclical monotonicity, a condition to characterize the subgradient correspondence for convex real-valued functions (see Rockafellar, 2015).

Definition A.5. $\mathcal{D}$ is *cyclically monotone* if for any sequence of observations $(m_\ell)_{\ell \in [L]}$

$$(p^{m_L} \cdot x^{m_1} - 1) + (p^{m_1} \cdot x^{m_2} - 1) + (p^{m_2} \cdot x^{m_3} - 1) + \ldots + \cdot (p^{m_{L-1}} x^{m_L} - 1) \geq 0\,. \tag{2}$$

If $\mathcal{D}$ satisfies cyclical monotonicity, the homothetic function $U(x) + y$ can always be chosen such that $U(\cdot)$ is continuous, strictly increasing, and convex. Furthermore, Brown and Calsamiglia (2007) show that this property is equivalent to the existence of numbers $u^i \in \mathbb{R}$ such that $u^i \geq u^j + 1 - p^i \cdot x^j$, which have the intuitive interpretation that a homothetic utility assures that the marginal utility of income, whose analogous in the Afriat inequalities is λ^i, is always equal to one.

As in the case of homothetic utilities and HARP, a quasilinear utility and cyclical monotonicity allow us to infer further indifferences than the ones in the revealed preferences. To see this, take the example $(p^i \cdot x^j - 1) + (p^j \cdot x^m - 1) + (p^m \cdot x^i - 1) = 0$, and suppose such choices are quasilinear rationalizable by $U(x) + y$. As x^i is optimal from p^i we have $p^i \cdot x^j - 1 \geq U(x^j) - U(x^i)$. Similarly $p^j \cdot x^m - 1 \geq U(x^m) - U(x^j)$, and $p^m \cdot x^i - 1 \geq U(x^i) - U(x^m)$. From the three inequalities, we obtain

$$\begin{aligned} 0 &= (p^i \cdot x^j - 1) + (p^j \cdot x^m - 1) + (p^m \cdot x^i - 1) \\ &\geq U(x^j) - U(x^i) + U(x^m) - U(x^j) + U(x^i) - U(x^m) \\ &= 0\,. \end{aligned}$$

Since the inequality in the previous equation has to be an equality, $p^i \cdot x^j - 1 = U(x^j) - U(x^i)$, $p^j \cdot x^m - 1 = U(x^m) - U(x^j)$, and $p^m \cdot x^i - 1 = U(x^i) - U(x^m)$. Thus, x^j is optimal at prices p^i, x^m is optimal at price p^j, and x^i is optimal at price p^m.

Furthermore, suppose U is differentiable. Since both x^i and x^j are optimal from p^i, we can infer that (for interior solutions) the MRS at both bundles is given by the price ratios of p^i; similarly, we can conclude that x^j and x^m have the same MRS and that x^m and x^i also do (all for interior solutions).

As in the previous sections, we focus on indifferences to obtain differentiability of the utility function. As explained in the previous example, we infer such indifferences through the sequences of cyclical monotonicity equal to zero, which we call Q0-sequences.

Definition A.6. A *Q0-sequence* is a sequence of unique observations $(m_\ell)_{\ell \in [L]}$ such that

$$(p^{m_L} \cdot x^{m_1} - 1) + (p^{m_1} \cdot x^{m_2} - 1) + (p^{m_2} \cdot x^{m_3} - 1) + \ldots + \cdot (p^{m_{L-1}} x^{m_L} - 1) \leq 0$$

As with H1-sequences in Section II, including a weak inequality instead of equality ensures that the data set modification finishes in a finite number of steps. Furthermore, if the previous inequality is strict, we already know that the data set is not quasilinear rationalizable.

We define the quasilinear modification similarly to the homothetic one (Definition A.3). The primary difference is that prices do not need to be scaled in this case, as the budget constraint is nonbinding.

Definition A.7. For any Q0-sequence $s = (m_\ell)_{\ell \in [L]}$ let $I^Q(s) = \{p^{m_\ell}\}_{\ell \in [L]}$. Furthermore, denote by $\mathcal{S}^Q(i)$ the set of all Q0-sequences ending in i. The *quasilinear modification* of $\mathcal{D}$, $\Gamma^Q(\mathcal{D})$, is $\Gamma^Q(\mathcal{D}) = \left(r^{i,Q}, x^i\right)_{i \in [N]}$, where

$$r^{i,Q} = \bigwedge \bigcup_{s \in \mathcal{S}^Q(i)} I^Q(s)\,.$$

Every Q0-sequence s ending in i yields information about bundles that are indifferent to x^i; the previous definition collects all such bundles in $I^Q(s)$ and then collects all the bundles from the different sequences (by taking the union of the $I^Q(s)$s for different Q0-sequences finishing in i). This process yields all the bundles revealed indifferent to x^i.[8] We take the set of all observations whose choices we can infer are indifferent to x^i. Then, the quasilinear modification replaces the price p^i by the meet of prices in such a set. Since quasilinear rationalization is a different notion of rationalization, we need a result analogous to Proposition 1 for this notion.

Proposition A.4. *Let U be well-behaved and differentiable. $\mathcal{D}$ is quasilinear rationalizable by $U(x) + y$ if, and only if, $\Gamma^Q(\mathcal{D})$ also is.*

The following result is the equivalent to Proposition 2 for quasilinear utilities.

Proposition A.5. *Suppose $\mathcal{D} = \Gamma^Q(\mathcal{D})$. Then $\mathcal{D}$ is quasilinear rationalizable if, and only if, it is quasilinear rationalizable by $U(x)+y$, where U is infinitely differentiable.*

Our final result provides a characterization and test for quasilinear rationalization by $U(x) + y$ when U is differentiable. The proof proceeds similarly as in the previous theorems.

Theorem A.3. *Let $\mathcal{D}^Q_\wedge$ be the fixed point of Γ^Q, starting from $\mathcal{D}$. The following are equivalent:*

Q-1) $\mathcal{D}$ is quasilinear rationalizable by $U(x) + y$ and U is differentiable.

Q-2) $\mathcal{D}^Q_\wedge$ is quasilinear ratinoalizable (i.e., is cyclically monotone).

Q-3) For two bundles x^i, x^j, denote by $x^i \approx^Q_\wedge x^j$ the existence of a Q0-sequence $\mathcal{D}^Q_\wedge$ containing both i and j. There are numbers $u^i > 0$ and $\mu^i \geq \mathbf{0}$ such that

$$u^i > u^j + 1 - p^i \cdot x^j + \mu^i \cdot x^j \qquad \text{whenever } x^i \not\approx^Q x^j \tag{Q0}$$

$$u^i = u^j + 1 - p^i \cdot x^j + \mu^i \cdot x^j \qquad \text{whenever } x^i \approx^Q x^j \tag{Q2}$$

$$p^i - \mu^i = p^j - \mu^j \qquad \text{whenever } x^i \approx^Q x^j \tag{Q3}$$

[8] Note that the indifferences inferred from Q0-sequences include the ones inferred by the revealed indifferent relation presented in Definition 1.

$$p^i - \mu^i \gg \mathbf{0} \qquad \textit{for all } i \in [N] \tag{Q4}$$

$$\mu^i \cdot x^i = 0 \qquad \textit{for all } i \in [N] \tag{Q5}$$

Q-4) $\mathcal{D}$ is quasilinear rationalizable by $U(x) + y$ and U is infinitely differentiable.

IV Proofs

IV.I Proofs of Section I

Proof of Lemma A.1. Denote by $\sim_p$ the revealed indifferences in $\mathcal{D}$, and $\sim_r$ the ones in $\Gamma(\mathcal{D}) = (r^i, x^i)_{i\in[N]}$. Since $x^i = x^j \implies x^i \sim_p x^j$, we have $x^i = x^j \implies r^i = r^j$. Since $\Gamma(\mathcal{D})$ is rationalizable by a strictly concave utility, it satisfies SARP. Hence $x^i \neq x^j \implies x^i \not\sim_r x^j$, and $\Gamma(\Gamma(\mathcal{D})) = \Gamma(\mathcal{D})$. □

Proof of Proposition A.1. Sufficiency is immediate. For necessity suppose $\mathcal{D}$ is rationalizable by a strictly concave utility. Since $\mathcal{D}$ satisfies SARP, it satisfies GARP and $x^i \neq x^j \implies x^i \not\sim x^j$. By Lemma 3 there are numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ such that $x^i \neq x^j$ implies $u^i > u^j + \lambda^i(1 - p^i \cdot x^j)$ and $x^i = x^j$ implies $u^i = u^j$ and $\lambda^i = \lambda^j$.

Set $M > 0$ and define the function $g : \mathbb{R}^K \to \mathbb{R}_+$ by $g(x) = \left(M + ||x||^2\right)^{1/2} - M^{1/2}$, which is continuous, strictly convex, and satisfies $g(\mathbf{0}) = 0$, and $g(x) > 0$ for $x \neq \mathbf{0}$ (see Matzkin & Richter, 1991). As $N < \infty$ and $\lambda^i p^i_k > 0$ for all $i \in [N]$ and $k \in [K]$, there is $\varepsilon > 0$ small enough such that

$$u^i - \varepsilon g(x^i - x^j) > u^j + \lambda^i(1 - p^i x^j) \qquad \text{whenever } x^i \neq x^j\text{; and} \tag{3}$$

$$\lambda^i p^i_k > \varepsilon \qquad \text{for all } i \in [N], k \in [K]\,. \tag{4}$$

For each $i \in [N]$ define the function $\phi_i : \mathbb{R}^K \to \mathbb{R}$ by $\phi^i(x) = u^i - \lambda^i(1 - p^i \cdot x) - \varepsilon g(x - x^i)$, which is continuous, strictly concave, and strictly increasing.[9] Furthermore,

[9] $\phi^i(\cdot)$ is strictly increasing as for all $k \in [K]$ we have ${}^{\partial \phi^i(x)}/_{\partial x_k} = \lambda^i p^i_k - \mu^i_k -$

since $\mathcal{D} = \Gamma(\mathcal{D})$ we have $x^i = x^j \implies p^i = p^j$. Hence $x^i = x^j$ implies $\phi^i(x) = \phi^j(x)$ for all x.

Define $V : \mathbb{R}^K \to \mathbb{R}$ by

$$V(x) = \min_{i \in [N]} \phi^i(x) \,.$$

Since $V(\cdot)$ is the minimum of finitely many functions and all of them are continuous, strictly increasing, and strictly convex, it inherits these three properties. Furthermore, from (3) it follows that $\phi(x^i) < \phi^j(x^i)$ whenever $x^i \neq x^j$. Together with the fact that $\phi^i = \phi^j$ whenever $x^i = x^j$, this implies that $V(x^i) = u^i$ and $V(x) = \phi^i(x)$ in a neighborhood of x^i for every $i \in [N]$. Hence, there is $\eta > 0$ small enough such that for all $i \in [N]$ and $\xi \in B(\eta)$ we have $V(x^i - \xi) = \phi^i(x^i - \xi)$. Define $\tilde{U}(x) = (V \star \rho_\eta)(x)$, where ρ_η is the function defined in the Proof of Proposition 2 (Appendix C), but using the value of η defined here. Then $\tilde{U}$ is continuous, infinitely differentiable, strictly concave and strictly increasing.

For every $i \in [N]$ we have

$$\begin{aligned}
\tilde{U}(x^i) &= \int_{B(\eta)} \left[\min_{j \in [N]} \phi^j(x^i - \xi) \right] \rho_\eta(\xi)\, d\xi \\
&= \int_{B(\eta)} \phi^i(x^i - \xi) \rho_\eta(\xi)\, d\xi \\
&= \int_{B(\eta)} \left[u^i - \lambda^i(1 - p^i \cdot (x^i - \xi)) - \varepsilon g(-\xi) \right] \rho_\eta(\xi)\, d\xi \\
&= \left[u^i - \lambda^i(1 - p^i \cdot x^i) \right] \int_{B(\eta)} \rho_\eta(\xi)\, d\xi - \lambda^i\, p^i \cdot \int_{B(\eta)} \xi \rho_\eta(\xi)\, d\xi - \\
&\qquad - \varepsilon \int_{B(\eta)} g(-\xi) \rho_\eta(\xi)\, d\xi \\
&= u^i - \varepsilon \int_{B(\eta)} g(\xi) \rho_\eta(\xi)\, d\xi
\end{aligned} \tag{5}$$

The last equality follows from $p^i \cdot x^i = 1$, $\int_{B(\eta)} \rho_\eta(\xi) d\xi = 1$, $\int_{B(\eta)} \xi \rho_\eta(\xi) d\xi = \mathbf{0}$, and $g(x) = g(-x)$.

$\left(g(x) + M^{1/2} \right)^{-1} \varepsilon x_k > \lambda^i p^i_k - \mu^i_k - \varepsilon > 0$. The last inequality follows from (4).

Take x satisfying $x \neq x^i$ and $p^i \cdot x \leq 1$. Then

$$\begin{aligned}
\tilde{U}(x) &= \int_{B(\eta)} \left[\min_{j \in [N]} \phi^j(x - \xi) \right] \rho_\eta(\xi)\, d\xi \\
&= \int_{B(\eta)} \left[u^i - \lambda^i (1 - p^i \cdot (x - \xi)) - \varepsilon g(x - \xi - x^i) \right] \rho_\eta(\xi)\, d\xi \\
&= \left[u^i - \lambda^i (1 - p^i \cdot x) \right] \int_{B(\eta)} \rho_\eta(\xi)\, d\xi - \lambda^i p^i \cdot \int_{B(\eta)} \xi \rho_\eta(\xi)\, d\xi - \\
&\qquad - \varepsilon \int_{B(\eta)} g(x - \xi - x^i) \rho_\eta(\xi)\, d\xi \\
&\leq u^i - \varepsilon \int_{B(\eta)} g((x - x^i) - \xi) \rho_\eta(\xi)\, d\xi \\
&= u^i - \varepsilon \int_{B(\eta)} g(\xi + (x^i - x)) \rho_\eta(\xi)\, d\xi \\
&< u^i - \varepsilon \int_{B(\eta)} \left[g(\xi) + \nabla g(\xi) \cdot (x - x^i) \right] \rho_\eta(\xi)\, d\xi \\
&= \tilde{U}(x^i) - \varepsilon (x - x^i) \cdot \int_{B(\eta)} \xi \frac{\rho_\eta(\xi)}{(M + ||\xi||^2)^{1/2}}\, d\xi \\
&= \tilde{U}(x^i)
\end{aligned}$$

The second line follows from $i \in [N]$, and the definition of ϕ^i; the third one rearranges terms; the fourth one from $\int_{B(\eta)} \rho_\eta(\xi) d\xi = 1$, $\int_{B(\eta)} \xi \rho_\eta(\xi) d\xi = \mathbf{0}$, $p^i \cdot x \leq 1$, and $\lambda^i > 0$; the fifth one from symmetry of $g(x) = g(-x)$; the sixth one from the strict convexity of g;[10] the seventh one rearranges terms and replaces (5) and $\nabla g(\xi)$; and the last one from the fact that $\frac{\rho_\eta(\xi)}{(M + ||\xi||^2)^{1/2}}$ is symmetric around zero, thus the integral is equal to zero.

Finally, let U be the restriction of $\tilde{U}$ to $\mathbb{R}^K_+$. Then U is strictly increasing, strictly concave, infinitely differentiable, and rationalizes $\mathcal{D}$. $\square$

Proof of Theorem A.1. That Sa-1) implies Sa-2) follows from Proposition 1. To see that Sa-2) implies Sa-3) note that if $\Gamma(\mathcal{D})$ is rationalizable by a strictly concave utility then $\Gamma(\mathcal{D}) = \Gamma(\Gamma(\mathcal{D}))$ (Lemma A.1). Then we can take the numbers u^i and

[10] Strict convexity of g implies $g(x) + \nabla g(x) \cdot (x - y) > g(y)$ whenever $x \neq y$ (with equality if $x = y$). The inequality follows replacing $x = \xi$ and $y = \xi + (x^i - x)$.

λ^i from Lemma 3 applied to $\Gamma(\mathcal{D})$ and define $\mu^i = \lambda^i(p^i - r^i)$; it is straightforward to see that (u^i, λ^i, μ^i) satisfy all the conditions. Starting from Sa-3) we can construct an infinitely differentiable and strictly concave utility that smoothly rationalizes $\mathcal{D}$ by a construction similar to the one in the proof of Proposition A.1; the only differences are that in this case $\phi^i = u^i - \lambda^i(1 - p^i \cdot x) - \mu^i \cdot x - \varepsilon g(x - x^i)$ and that to show that $\tilde{U}(x) < \tilde{U}(x^i)$ whenever $p^i \cdot x \leq 1$ we need to assume $x \geq \mathbf{0}$ (which is inconsequential for the purposes of our proof). That Sa-4) implies Sa-1) is immediate. □

IV.II Proofs of Section II

Lemma A.2. *Let $\approx^H$ be defined by $x^i \approx^H x^j$ if there is an H1-sequence containing i and j. If $\mathcal{D} = \Gamma^H(\mathcal{D})$ and $\mathcal{D}$ is rationalizable by a homothetic utility, then $x^i \approx^H x^j$ if, and only if, $(p^i \cdot x^j)(p^j \cdot x^i) = 1$.*

Proof. Since $\mathcal{D}$ is rationalizable by a homothetic utility it satisfies HARP (Varian, 1983), hence any H1-sequence is equal to one. Necessity follows from the definition of an H1-sequence (Definition A.2). For necessity suppose $x^i \approx^H x^j$; then there is an H1-sequence $(m_\ell)_{\ell \in [L]}$ such that $\ell' = i$ and (without loss of generality) $L = j$. By definition of Γ^H, and since $\mathcal{D} = \Gamma^H(\mathcal{D})$, we have

$$p^j \leq \left(\prod_{\ell=\ell'}^{L-1} p^{m_\ell} \cdot x^{m_{\ell+1}} \right)^{-1} p^i \,.$$

Since $p^i \cdot x^i = 1$,

$$p^j \cdot x^i \leq \left(\prod_{\ell=\ell'}^{L-1} p^{m_\ell} \cdot x^{m_{\ell+1}} \right)^{-1} \tag{6}$$

Similarly, by definition of Γ^H, and since $\mathcal{D} = \Gamma^H(\mathcal{D})$, we have

$$p^i \leq \left((p^j \cdot x^{m_1}) \prod_{\ell=1}^{\ell'-1} p^{m_\ell} \cdot x^{m_{\ell+1}} \right)^{-1} p^j \, ;$$

which, as $p^j \cdot x^j = 1$, implies

$$p^i \cdot x^j \leq \left((p^j \cdot x^{m_1}) \prod_{\ell=1}^{\ell'-1} p^{m_\ell} \cdot x^{m_{\ell+1}} \right)^{-1} .$$

Multiplying the previous equation and (6) we get

$$(p^i \cdot x^j)(p^j \cdot x^i) \leq \left((p^j \cdot x^{m_1}) \prod_{\ell=1}^{L-1} p^{m_\ell} \cdot x^{m_{\ell+1}} \right)^{-1} .$$

Since $(m_\ell)_{\ell \in [L]}$ is an H1-sequence and $j = L$, the last equation implies $(p^i \cdot x^j)(p^j \cdot x^i) \leq 1$. Finally, since $\mathcal{D}$ satisfies HARP we have $(p^i \cdot x^j)(p^j \cdot x^i) \geq 1$, which implies the desired result. $\square$

Lemma A.3. *Let $\mathcal{Z}(i)$ be the set of all finite sequences of observations $(m_\ell)_{\ell \in [L]}$ satisfying $x^{m_L} = x^i$. Define*

$$v^i = \min_{\mathcal{Z}(i)} (p^{m_1} \cdot x^{m_2})(p^{m_2} \cdot x^{m_3}) \dots (p^{m_{L-1}} \cdot x^i)$$

If $\mathcal{D}$ satisfies HARP and $\mathcal{D} = \Gamma^H(\mathcal{D})$, then $v^i \leq v^j \, p^j \cdot x^i$ for all $i, j \in [N]$, $v^i = v^j \, p^j \cdot x^i$ whenever $x^i \approx^H x^j$, and there is $i' \in [N]$ such that $v^j < v^m p^m \cdot x^j$ whenever $x^{i'} \approx^H x^j$ and $x^{i'} \not\approx^H x^m$.

Proof. As $\mathcal{D}$ satisfies HARP, by the proof of Theorem 2 in Varian (1983) we know that v^i is well defined; this is, that the minimum is achieved. Furthermore, it follows from the same proof that $v^i \leq v^j \, p^j \cdot x^i$ for all i, j. Since $p^i \gg \mathbf{0}$ and $x^i > \mathbf{0}$ for all i, we have $v^i > 0$.

First we show that $v^i = v^j p^j \cdot x^i$ whenever $x^i \approx^H x^j$. By Lemma A.2, $x^i \approx^H x^j$

implies $(p^j \cdot x^i)(p^i \cdot x^j) = 1$. As $v^i \le v^j\, p^j \cdot x^i$ and $v^j \le v^i\, p^i \cdot x^j$ we have

$$v^i \le v^j \, \frac{(p^i \cdot x^j)(p^j \cdot x^i)}{p^i \cdot x^j} = v^j \, \frac{1}{p^i \cdot x^j} \le v^i \, .$$

Therefore all the inequalities have to be equalities; in particular $v^i\, p^i \cdot x^j = v^j$.

We show that there is $i' \in [N]$ such that $i' \in [N]$ such that $v^j < v^m p^m \cdot x^j$ whenever $x^{i'} \approx^H x^j$ and $x^{i'} \not\approx^H x^m$ by contradiction. Suppose for every $i \in [N]$ there are $x^j \approx^H x^i$ and $x^m \not\approx^H x^i$ such that $v^j = v^m\, p^m \cdot x^j$. Since $x^j \approx^H x^i$, Lemma A.2 implies $(p^i \cdot x^j)(p^j \cdot x^i) = 1$. Hence $v^m\,(p^m \cdot x^j)(p^j \cdot x^i) = v^j (p^j \cdot x^i) = v^i (p^i \cdot x^j)(p^j \cdot x^i) = v^i$. Thus we can construct an infinite sequence $(x^{m_\ell})_{\ell \in [L]}$ such that, for all ℓ, $x^{m_\ell} \approx^H x^{m_{\ell+1}}$ and $x^{m_\ell} \not\approx^H x^{m_{\ell+2}}$, and satisfies

$$v^{m_1} = v^{m_2}(p^{m_2} \cdot x^{m_1}) = v^{m_3}(p^{m_3} \cdot x^{m_2})(p^{m_2} \cdot x^{m_1}) = \ldots = v^{m_\ell} \prod_{s=1}^{\ell-1} p^{m_{s+1}} \cdot x^{m_s} = \ldots$$

Since the sequence is infinite and there are finitely many observations, there have to be ℓ', ℓ'', with $\ell' + 2 < \ell''$ such that $m_{\ell'} = m_{\ell''}$. Then

$$v^{m_{\ell'}} \prod_{\ell=1}^{\ell'-1} p^{m_{\ell+1}} \cdot x^{m_\ell} = v^{m_{\ell''}} \prod_{\ell=1}^{\ell''-1} p^{m_{\ell+1}} \cdot x^{m_\ell} = v^{m_{\ell''}} \left(\prod_{\ell=1}^{\ell'-1} p^{m_{\ell+1}} \cdot x^{m_\ell} \right) \left(\prod_{\ell=\ell'}^{\ell''-1} p^{m_{\ell+1}} \cdot x^{m_\ell} \right) .$$

Since $v^{m_{\ell'}} = v^{m_{\ell''}}$ and $p^{m'_\ell} = p^{m''_\ell}$ we have

$$1 = \prod_{\ell=\ell'}^{\ell''-1} p^{m_{\ell+1}} \cdot x^{m_\ell} = (p^{m_{\ell'+1}} \cdot x^{m_{\ell'}})(p^{m_{\ell'+2}} \cdot x^{m_{\ell'+1}}) \ldots (p^{m_{\ell'}} \cdot x^{m_{\ell''-1}}) \, .$$

Therefore $x^{m'_\ell} \approx^H x^{m_{\ell'+2}}$, a contradiction. □

Lemma A.4. *If $\mathcal{D}$ satisfies HARP and $\mathcal{D} = \Gamma^H(\mathcal{D})$ then there are numbers $u^i > 0$ such that $u^i\, p^i \cdot x^j > u^j$ whenever $x^i \not\approx^H x^j$ and $u^i\, p^i \cdot x^j = u^j$ whenever $x^i \approx^H x^j$.*

Proof. We show the result by induction of N. If $N = 1$ then $u^1 = 1$ satisfies the conditions.

Suppose the conditions hold for any database of $N-1$ or less elements, and take $\mathcal{D}$ comprised of N observations. Take the numbers v^i defined in Lemma A.3 and i' such that $v^j < v^m p^m \cdot x^j$ whenever $x^{i'} \approx^H x^j$ and $x^{i'} \not\approx^H x^m$. Denote $E = \{i \in [N] : x^i \approx^H x^{i'}\}$ and $D = [N] \setminus E$. If $D = \emptyset$, Lemma A.3 imply that the numbers v^i satisfy the conditions.

If $D \neq \emptyset$, then $(p^i, x^i)_{i \in D}$ is a data set comprised of $N-1$ or less observations. By induction hypothesis there are numbers $\tilde{u}^m > 0$ for $m \in D$ such that the conditions hold. Take the numbers v^i defined in Lemma A.3 and define

$$
\begin{aligned}
\alpha &= \min_{i \in I} \min_{m \in D} \frac{v^m \, p^m \cdot x^i}{v^i} - 1 > 0 \\
u^i &= \left(1 + \frac{\alpha}{2}\right) v^i && \text{for every } i \in E\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
&u^i \, p^i \cdot x^m > v^m && \text{whenever } i \in E \text{ and } m \in D, && (7)\\
&v^m \, p^m \cdot x^i > u^i && \text{whenever } i \in E \text{ and } m \in D, && (8)\\
&u^i \, p^i \cdot x^j = u^j && \text{whenever } i, j \in E\,. && (9)
\end{aligned}
$$

Take a sequence $\beta^n \to 1$, where $\beta^n \in (0,1)$ for all n, and for each $m \in D$ and $n \in \mathbb{N}$ define $w^m(n) = (v^m)^{\beta^n} \, (\tilde{u}^m)^{1-\beta^n}$. Take any $m, m' \in D$ and $n \in \mathbb{N}$. If $x^m \approx^H x^{m'}$

$$
w^m(n) \, p^m \cdot x^{m'} = (v^m \, p^m \cdot x^{m'})^{\beta^n} (\tilde{u}^m \, p^m \cdot x^{m'})^{1-\beta^n} = (v^{m'})^{\beta^n} \, (\tilde{u}^{m'})^{1-\beta^n} = w^{m'}(n)\,;
$$

and if $x^m \not\approx^H x^{m'}$

$$
w^m(n) \, p^m \cdot x^{m'} = (v^m \, p^m \cdot x^{m'})^{\beta^n} (\tilde{u}^m \, p^m \cdot x^{m'})^{1-\beta^n} > (v^{m'})^{\beta^n} \, (\tilde{u}^{m'})^{1-\beta^n} = w^{m'}(n)\,.
$$

Since $w^m(n) \to v^m$ for every $m \in \mathcal{D}$, for n_0 large enough we have

$$\left.\begin{aligned} &u^i\, p^i \cdot x^m > w^m(n_0) \\ &w^m(n_0)\, p^m \cdot x^i > u^i \end{aligned}\right. \qquad \text{whenever } i \in E \text{ and } m \in D\,.$$

Setting $u^m = w^m(n_0)$ for all $m \in D$ assures that the numbers u^i satisfy the desired properties. □

Lemma A.5. *If $\mathcal{D}$ satisfies HARP and $\mathcal{D} = \Gamma^H(\mathcal{D})$, there exist an open set S and a strictly increasing, concave, and infinitely differentiable function $f : S \to \mathbb{R}$, satisfying the following properties:*

1. $\left(\mathbb{R}^K_+ \setminus \{\mathbf{0}\}\right) \subset S$;
2. *For every $x \in S$ there is a unique value $\alpha(x) > 0$ satisfying $f(\alpha(x)^{-1}x) = 1$.*
3. $f\left({x^i}/{f(x^i)}\right) \geq f(y)$ *whenever and* $p^i \cdot y \leq {1}/{f(x^i)}\, p^i \cdot x^i$

Proof. Take $u^i > 0$ from Lemma A.4 and define $\phi^i(x) = u^i p^i \cdot x$, and $V(x) = \min_{i\in[N]} \phi^i(x)$. Since ϕ^i is strictly increasing, concave and homogeneous of degree one, V inherits all these properties. Furthermore, $V(x) > 0$ whenever $x > \mathbf{0}$.

By Lemma A.4, $\phi^i(x) = \phi^j(x)$ for all x whenever $x^i \approx^H x^j$, and $V(x^i) = u^i$. Furthermore, there is κ small enough such that, for every $\xi \in B(\kappa)$, $\phi^i(x^i - \xi) = \phi^j(x^i - \xi)$ whenever $x^i \approx^H x^j$ and $\phi^i(x^i - \xi) < \phi^j(x^i - \xi)$ whenever $x^i \not\approx^H x^j$.

For every $i \in [N]$ define $y^i = V(x^i)^{-1}x^i$, and note that $V(y^i) = \phi^i(y^i) = 1$. Let $\overline{v} = \max_{i\in[N]} V(x^i) > 0$ and let $\eta = \kappa \min\{1, {1}/{\overline{v}}\} > 0$. Define $\tilde{U}(x) = (V \star \rho_\eta)(x)$, where ρ_η is the function defined in the Proof of Proposition 2 (Appendix C), but using the value of η defined here. Then $\tilde{U}$ is strictly increasing, concave, and infinitely differentiable.

Let

$$S = \bigcap_{i\in[N]} \left\{x \in \mathbb{R}^K_+ : \phi^i(x) > 0\right\}\,.$$

Since each function ϕ^i is affine, the sets $\left\{x \in \mathbb{R}^K_+ : \phi^i(x) > 0\right\}$ are open, hence S also is.

Let $f : S \to \mathbb{R}$ be the restriction of $\tilde{U}$ to S. We now show the required properties, following the same enumeration as the statement of the Lemma.

1. From (H4), for every $x > \mathbf{0}$ we have $\phi^i(x) > 0$ for all $i \in [N]$. Therefore $\left(\mathbb{R}^K_+ \setminus \{\mathbf{0}\}\right) \subset S$.

2. Take an arbitrary $x \in S$; we show that there is a unique $\alpha(x)$ such that $f(\alpha(x)^1 x) = 1$. Since $x \in S$, we have $\phi^i(x) = u^i\, p^i \cdot x > 0$ for all $i \in [N]$, therefore (as $[N]$ is finite) $\min_{i\in[N]} u^i\, p^i \cdot x > 0$. Hence for $\beta > 0$ small enough

$$\beta^{-1} \min_{i\in[N]} u^i\, p^i \cdot x > 1 + \max_{j\in[N]} u^j\, p^j \cdot (\eta \mathbf{1}) \tag{10}$$

Thus

$$\begin{aligned}
f(\beta^{-1}x) &= \int_{B(\eta)} \min_{i\in[N]} u^i\, p^i \cdot (\beta^{-1}x - \xi) \rho_\eta(\xi)\, d\xi \\
&\geq \int_{B(\eta)} \left(\beta^{-1} \min_{i\in[N]} u^i\, p^i \cdot x - \max_{j\in[N]} u^j\, p^j \cdot \xi \right) \rho_\eta(\xi)\, d\xi \\
&\geq \int_{B(\eta)} \left(\beta^{-1} \min_{i\in[N]} u^i\, p^i \cdot x - \max_{j\in[N]} u^j\, p^j \cdot (\eta \mathbf{1}) \right) \rho_\eta(\xi)\, d\xi \\
&> \int_{B(\eta)} \rho_\eta(\xi)\, d\xi \\
&= 1
\end{aligned}$$

The first inequality splits terms; the second one follows from $\xi \in B(\eta)$; the third one from (10); and the last one replaces $\int_{B(\eta)} \rho_\eta(\xi)\, d\xi = 1$. Fix $i \in [N]$. Since $\tilde{U}$ is continuous

$$\begin{aligned}
\lim_{\beta\to\infty} f(\beta^{-1}x) &= \lim_{\beta\to\infty} \tilde{U}(\beta^{-1}y) \\
&= \tilde{U}(\mathbf{0})
\end{aligned}$$

$$
\begin{aligned}
&= \int_{B(\eta)} \min_{j\in[N]} \phi^j(\mathbf{0}-\xi)\rho_\eta(\xi)\,d\xi \\
&\le \int_{B(\eta)} \phi^i(\mathbf{0}-\xi)\rho_\eta(\xi)\,d\xi \\
&= u^i\,p^i\cdot\left(\mathbf{0}\int_{B(\eta)} \rho_\eta(\xi)\,d\xi - \int_{B(\eta)} \xi\rho_\eta(\xi)\,d\xi\right) \\
&= 0
\end{aligned}
$$

Since f inherits continuity from $\tilde{U}$ and $\lim_{\beta\to\infty}\beta^{-1}x = \mathbf{0}$, the Intermediate Value Theorem implies that there is $\alpha(x)\in(\beta,\infty)$ satisfying $f(\alpha(x)^{-1}x)=1$. We show that $\alpha(x)$ is unique by showing that $f(\alpha x)$ is strictly increasing in α. Take $\gamma>\alpha>0$; then

$$
\begin{aligned}
f(\gamma x)-f(\alpha x) &= \int_{B(\eta)} \min_{j\in[N]} \phi^j(\gamma x-\xi)\rho_\eta(\xi)\,d\xi - \int_{B(\eta)} \min_{j\in[N]} \phi^j(\alpha x-\xi)\rho_\eta(\xi)\,d\xi \\
&= \int_{B(\eta)} \left(\min_{j\in[N]} \phi^j(\gamma x-\xi) - \min_{j\in[N]} \phi^j(\alpha x-\xi)\right)\rho_\eta(\xi)\,d\xi \\
&\ge \int_{B(\eta)} \left(\min_{j\in[N]} \phi^j(\gamma x-\xi) - \phi^j(\alpha x-\xi)\right)\rho_\eta(\xi)\,d\xi \\
&= (\gamma-\alpha)\int_{B(\eta)} \min_{j\in[N]} u^i\,p^i\cdot x\,\rho_\eta(\xi)\,d\xi \\
&> 0\,.
\end{aligned}
$$

where the first inequality follows from $\min f(x)-\min g(x)\ge\min(f(x)-g(x))$,[11] and the last one from $\gamma>\alpha$ and $x\in S$.

3. As $\eta\le\kappa$, $V(x^i-\xi)=\phi^i(x^i-\xi)$ whenever $\xi\in B(\eta)$. Hence

$$
\begin{aligned}
f(x^i) &= \int_{B(\eta)} V(x^i-\xi)\rho_\eta(\xi)\,d\xi \\
&= \int_{B(\eta)} \phi^i(x^i-\xi)\rho_\eta(\xi)\,d\xi
\end{aligned}
$$

[11] Let x_0 be the minimizer of $f(x)$. Then $\min f(x)-\min g(x)\ge\min f(x)-g(x_0)=f(x_0)-g(x_0)\ge\min(f(x)-g(x))$.

$$
\begin{aligned}
&= \int_{B(\eta)} u^i\, p^i \cdot (x^i - \xi)\rho_\eta(\xi)\, d\xi \\
&= u^i\, p^i \cdot \left(x^i \int_{B(\eta)} \rho_\eta(\xi)\, d\xi - \int_{B(\eta)} \xi \rho_\eta(\xi)\, d\xi \right) \\
&= u^i\, p^i \cdot x^i \\
&= u^i
\end{aligned}
$$

Therefore ${x^i}/{f(x^i)} = y^i$. In a similar way, as $\eta \leq {\kappa}/{\overline{v}} \leq {\kappa}/{V(x^i)}$, whenever $\xi \in B(\eta)$ we have $V(x^i)\xi \in B(V(x^i)\eta) \subset B(\kappa)$, and

$$
\begin{aligned}
V(y^i - \xi) &= \frac{1}{V(x^i)} V(V(x^i)y^i - V(x^i)\xi) \\
&= \frac{1}{V(x^i)} V(x^i - V(x^i)\xi) \\
&= \frac{1}{V(x^i)} \phi^i(x^i - V(x^i)\xi) \\
&= \frac{1}{V(x^i)} \phi^i(V(x^i)y^i - V(x^i)\xi) \\
&= \phi^i(y^i - \xi)\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
f(y^i) &= \int_{B(\eta)} V(y^i - \xi)\rho_\eta(\xi)\, d\xi \\
&= u^i\, p^i \cdot \left(y^i \int_{B(\eta)} \rho_\eta(\xi)\, d\xi - \int_{B(\eta)} \xi \rho_\eta(\xi)\, d\xi \right) \\
&= u^i\, p^i \cdot y^i \\
&= 1\,.
\end{aligned}
$$

Finally, take y such that $p^i \cdot y \leq p^i \cdot y^i$. We have

$$
f(y) = \int_{B(\eta)} \min_{j \in [N]} \phi^j(y - \xi)\rho_\eta(\xi)\, d\xi
$$

$$
\begin{aligned}
&\leq \int_{B(\eta)} u^i\, p^i \cdot (y - \xi)\, \rho_\eta(\xi)\, d\xi \\
&= u^i\, p^i \cdot \left(y \int_{B(\eta)} \rho_\eta(\xi)\, d\xi - \int_{B(\eta)} \xi \rho_\eta(\xi)\, d\xi \right) \\
&\leq u^i\, p^i \cdot y^i \\
&= 1 \\
&= f(y^i)
\end{aligned}
$$

This completes the proof.

$\square$

Proof of Proposition A.3. Suffieincy is immediate. For necessity take the open set S and the function $f : S \to \mathbb{R}$ from Lemma A.5. For every $x \in S$ take the unique value $\alpha(x) > 0$ such that $f(\alpha(x)^{-1}x) = 1$ and define $W : S \cup \{\mathbf{0}\} \to \mathbb{R}$ by

$$
W(x) = \begin{cases} \alpha(x) & \text{if } x \in S \\ 0 & \text{if } x = \mathbf{0} \end{cases} \tag{11}
$$

From the previous definition we have $W(x) > 0$ whenever $x \in S$. Furthermore, from the proof of Lemma A.5 we have that $f(\alpha x)$ is strictly increasing in α. Hence both f and W have the same upper and lower contour sets around one; this is, $W(x) \geq 1 \iff f(x) \geq 1$, and $W(x) > 1 \iff f(x) > 1$.

Now we show that W is continuous, strictly increasing, concave, infinitely differentiable in S, homothetic, and rationalizes the data.

- W is homothetic: Take $\lambda > 0$. Since $W(\mathbf{0}) = 0$ we have $\lambda W(\mathbf{0}) = W(\lambda \mathbf{0})$. If $x \neq \mathbf{0}$ we have $f(W(x)^{-1}x) = 1$, and $1 = f(W(\lambda x)^{-1}\lambda x)$. Since there is a unique value $\alpha(x) > 0$ such that $f(\alpha(x)^{-1}x) = 1$, we have $\alpha(x) = W(x)^{-1} = W(\lambda x)^{-1}\lambda$, therefore $\lambda W(x) = W(\lambda x)$.

- W is strictly increasing: Take x, y such that $x > y$. If $y = \mathbf{0}$ by definition of W we have $W(x) > 0 = W(y)$. If $y \neq \mathbf{0}$, towards a contradiction suppose $U(y) \geq U(x) > 0$, which implies $W(x)^{-1}x > W(y)^{-1}y$. As f is strictly increasing, we have $f(W(y)^{-1}y) < f(W(x)^{-1}x)$. But the definition of W implies $f(W(y)^{-1}y) = f(W(x)^{-1}x) = 1$, a contradiction.

- W is concave: Denote by $P_a^f = \{x \in \mathbb{R}_+^K : f(x) \geq a\}$ and $P_a^W = \{x \in \mathbb{R}_+^K : W(x) \geq a\}$ the superlevel sets of f and W at a, respectively. Since both W and f have the same contour sets around one, we have $P_1^f = P_1^W$. Since f is concave, P_1^f is convex, and P_1^W also is. Furthermore, homotheticity of W implies that for any $a \geq 0$ we have $P_a^W = \{a\, x \in \mathbb{R}_+^K : x \in P_1^W\}$, hence P_a^W is convex as well. This implies that the epigraph of W is convex, therefore W is concave.

- W is continuous: For continuity at $\mathbf{0}$, take $x^n \to \mathbf{0}$ ($x^n \in S$) and $\delta > 0$, and denote by $\mathbf{1}$ the K-dimensional vector with all components equal to one. Since $x^n \to \mathbf{0}$, for n large enough we have $x_k^m < W(\mathbf{1})^{-1}\delta$ for all $k \in [K]$ and $m \geq n$. Thus, as f is strictly increasing, $f(\delta^{-1}x^m) < f(W(\mathbf{1})^{-1}\mathbf{1}) = 1$. Homogeneity of degree one implies $W(x^m) < \delta$. Since $W(x^n) \geq 0$ for all n and δ is arbitrary, we have $\lim_{n\to\infty} W(x^n) = W(\mathbf{0}) = 0$.

  To see that W is continuous on S take a, b such that $0 < a < b$. Since $W(x) > 0$ for all $x \in S$,

$$W^{-1}((a,b)) = W^{-1}((a,\infty) \cap (0,b)) = W^{-1}((a,\infty)) \cap W^{-1}((0,b))$$

  Furthermore

$$\begin{aligned} W^{-1}((a,\infty)) &= \{x \in S : W(x) > a\} \\ &= \left\{x \in S : f(a^{-1}\, x) < 1\right\} \\ &= \left\{a^{-1}\, y \in S : f(y) < 1\right\}. \end{aligned}$$

Since f is continuous, the set $\left\{y \in \mathbb{R}_+^K : f(y) < 1\right\}$ is open, hence $U^{-1}((a, \infty))$ also is. Similarly we can show that $U^{-1}((0, b))$ is open. As the intersection of two open sets is open, we conclude that W is continuous in S. Therefore W is continuous.

- W is infinitely differentiable in S: Denote by $I_a^f = \{x \in \mathbb{R}_+^K : Z(x) = a\}$ and $I_a^W = \{x \in \mathbb{R}_+^K : W(x) = a\}$ the level sets of f and W at level a, respectively. Since f and W have the same contour sets around one, we have $I_1^f = I_1^U$. Since f is infinitely differentiable, I_1^f is a $K-1$ dimensional C^∞ manifold (see Debreu, 1972), and I_1^W also is. Furthermore, as W is homogeneous of degree one, for every $a > 0$ we have $I_a^W = \{ax \in \mathbb{R}_+^K : x \in I_1^W\}$, hence I_a^W is also a $K-1$ dimensional C^∞ manifold. Since W is continuous, strictly increasing, homogeneous of degree one, and all its indifference sets in S are $K-1$ dimensional C^∞ manifolds, Theorem 1 in Neilson (1991) implies that W is infinitely differentiable in S.[12]

- $W(x)$ rationalizes $\mathcal{D}$: Take $i \in [N]$ and x satisfying $p^i \cdot x \leq 1$. Recall that $y^i = f(x^i)^{-1}x^i$. Since $f(y^i) = 1$, then $f(y) \leq 1$ whenever $p^i \cdot y \leq p^i \cdot y^i$. As f and W have the same contour sets around 1 and $f(y^i) = 1$, then $W(y) \leq 1$ whenever $p^i \cdot y \leq p^i \cdot y^i$. Furthermore, as $f(y^i) = 1$, we have $W(y^i) = 1$ as well. Hence as W is homothetic $W(x^i) = W(f(x^i)y^i) = f(x^i)W(y^i) = f(x^i)$ and whenever $p^i \cdot x \leq 1$

$$W(x) = f(x^i)W\left(\frac{p^i \cdot x^i}{f(x^i)}x\right) = f(x^i)W((p^i \cdot y^i)\, x) \leq f(x^i) = W(x^i)\,.$$

The first equality follows from W being homogeneous of degree one and $p^i \cdot x^i = 1$, the second from the definition of y^i, and the inequality from $p^i \cdot (p^i \cdot y^i)\, x \leq p^i \cdot y^i$, which implies $W((p^i \cdot y^i)\, x) \leq W(y^i) = 1$. Therefore W rationalizes $\mathcal{D}$.

[12] Although Theorem 1 in Neilson (1991) is developed for a function whose domain is $\mathbb{R}_{++}^K$, the proof does not uses anything particular about that domain and hence applies to every open domain.

Let $U : \mathbb{R}^K_+ \to \mathbb{R}$ the restriction of W to $\mathbb{R}^K_+$. Then U is continuous, strictly increasing, infinitely differentiable in $\mathbb{R}^K_+ \setminus \{\mathbf{0}\}$, homothetic, and rationalizes the data. ☐

Proof of Theorem A.2. That H-1) implies H-2) follows from an iterative application of Proposition 1. That H-2) implies H-3) follows from Lemma A.4 applied to $\mathcal{D}^H_\wedge = (q^{i,H}, x^i)_{i\in[N]}$, and defining $\mu^i = u^i(p^i - q^{i,H})$. Starting from H-3) we can construct an infinitely differentiable (everywhere except at $\mathbf{0}$) and homothetic utility that smoothly rationalizes $\mathcal{D}$ by a construction similar to the one in the proof of Proposition A.3; the only differences are that in this case $\phi^i = (u^i p^i - \mu^i) \cdot x$ and that property 3 in Lemma A.5 applies only for $y > \mathbf{0}$ (which is inconsequential for the purposes of our proof). Finally, that H-4) implies H-1) is immediate. ☐

IV.III Proofs of Section III

Lemma A.6. *Suppose $\mathcal{D}$ is quasilinear rationalizable by $U(x) + y$, where U is differentiable, and $(m_\ell)_{\ell\in[L]}$ is such that*

$$p^{m_L} \cdot (x^{m_1} - x^{m_L}) + \sum_{\ell=1}^{L-1} p^{m_\ell} \cdot (x^{m_\ell} - x^{m_{\ell+1}}) = 0\,. \tag{12}$$

Then $\nabla U(x^{m_1}) = \nabla U(x^{m_2}) = \ldots = \nabla U(x^{m_L})$.

Proof. Since $\mathcal{D}$ is quasilinear rationalizable by $U(x)+y$, we have $p^{m_\ell} \cdot (x^{m_{\ell+1}} - x^{m_\ell}) \geq U(x^{m_{\ell+1}}) - U(x^{m_\ell})$ for all $\ell \in [L-1]$, and $p^{m_L} \cdot (x^{m_1} - x^{m_L}) \geq U(x^{m_1}) - U(x^{m_L})$. Replacing into (12) we obtain

$$\begin{aligned}
0 &= p^{m_L} \cdot (x^{m_1} - x^{m_L}) + \sum_{\ell=1}^{L-1} p^{m_\ell} \cdot (x^{m_\ell+1} - x^{m_\ell}) \\
&\geq U(x^{m_1}) - U(x^{m_L}) + \sum_{\ell=1}^{L-1} U(x^{m_\ell+1}) - U(x^{m_\ell}) \\
&= 0\,.
\end{aligned}$$

Since the inequality of the previous equation has to be an equality, we conclude that $p^{m_\ell} \cdot (x^{m_{\ell+1}} - x^{m_\ell}) = U(x^{m_{\ell+1}}) - U(x^{m_\ell})$, i.e. $U(x^{m_{\ell+1}})$ is optimal at price p^ℓ, for all $\ell \in [L-1]$. Similarly, x^{m_1} is optimal from price p^{m_L}. Since U is differentiable, Theorem 2 in Jeyakumar et al. (2004) implies $\nabla U(x^{m_\ell}) = \nabla U(x^{m_{\ell+1}})$ for all $\ell \in [L-1]$ and $\nabla U(x^{m_L}) = \nabla U(x^{m_1})$. Therefore $\nabla U(x^{m_1}) = \nabla U(x^{m_2}) = \ldots = \nabla U(x^{m_L})$. $\square$

Proof of Proposition A.4. For necessity suppose $\mathcal{D}$ is quasilinear rationalizable by $U(x)+y$. From the first order conditions of the maximization problem, for every $i \in [N]$ we have $\nabla U(x^i) = p^i - \mu^i \leq p^i$, where $\mu^i \geq \mathbf{0}$. Furthermore, the complementary slackness conditions on μ^i imply $\mu^i \cdot x^i = 0$; therefore $\nabla U(x^i) \cdot x^i = p^i \cdot x^i = 1$.

Take $i \in [N]$ and $s = (m_\ell)_{\ell \in [L]} \in \mathcal{S}^Q(i)$ a Q0-sequence satisfying $m_L = i$. Since $\mathcal{D}$ is quasilinear rationalizable by U, then $\nabla U(x^{m_\ell}) = p^{m_\ell} + \mu^{m_\ell} \leq p^{m_\ell}$ for all $\ell \in [L]$. Furthermore, Lemma A.6 implies $\nabla U(x^i) = \nabla U(x^{m_\ell})$ for every $\ell \in [L]$; hence $\nabla U(x^i) \leq p$ for every $p \in I^Q(s)$. Thus $\nabla U(x^i) \leq \bigwedge I^Q(s)$. As s is an arbitrary sequence in $\mathcal{S}^Q(i)$ we conclude that

$$\nabla U(x^i) \leq \bigwedge_{s \in \mathcal{S}^Q(i)} \bigwedge I^Q(s) = \bigwedge \bigcup_{s \in \mathcal{S}^Q(i)} I^Q(s) = r^{i,Q}\,.$$

Define $f : \mathbb{R}^K_+ \times \mathbb{R} \to \mathbb{R}$ by $f((x,y)) = U(x) - y$. Since U is concave, f also is. Hence for every $i \in [N]$ and $x \in \mathbb{R}^K_+$

$$\begin{aligned} U(x^i) - r^{i,Q} \cdot x^i - \big(U(x) - r^{i,Q} \cdot x\big) &= f(x^i,\, r^{i,Q} \cdot x^i) - f(x,\, r^{i,Q} \cdot x) \\ &\geq \nabla f(x^i,\, r^{i,Q} \cdot x^i) \cdot \big((x^i,\, r^{i,Q} \cdot x^i) - (x,\, r^{i,Q} \cdot x)\big) \\ &= \nabla U(x^i) \cdot (x^i - x) - \big(r^{i,Q} \cdot x^i - r^{i,Q} \cdot x\big) \\ &= p^i \cdot x^i - \nabla U(x^i) \cdot x - r^{i,Q} \cdot x^i + r^{i,Q} \cdot x \\ &\geq p^i \cdot x^i - r^{i,Q} \cdot x - r^{i,Q} \cdot x^i + r^{i,Q} \cdot x \\ &= (p^i - r^{i,Q}) \cdot x^i \end{aligned}$$

$$\geq 0\,.$$

The first inequality follows form the concavity of f, the second from $\nabla U(x^i) \leq r^{i,Q}$ and $x \geq \mathbf{0}$, and the last one from $p^i \geq r^{i,Q}$ and $x^i > \mathbf{0}$. We conclude that $\Gamma^Q(\mathcal{D})$ is quasilinear rationalizable by $U(x) + y$.

For sufficiency suppose $\Gamma^Q(\mathcal{D})$ is quasilinear rationalizable by $U(x)+y$. For every $i \in [N]$, the Lagrangian for the maximization problem in $\mathcal{D}$ is $U(x)+y-\lambda(p^i \cdot x+y)+\mu \cdot x$. Take the values $(x^i, 1, p^i - \nabla U(x^i))_{i\in[N]}$. Since $U(x)+y$ quasilinear rationalizes $\Gamma^Q(\mathcal{D})$, it is easy to see that these values satisfy the first order conditions of the maximization problem in $\mathcal{D}$. Finally, as $U(x)$ is strictly increasing, differentiable, and concave, $U(x)+y$ also is. Therefore the first order conditions are sufficient, and $\mathcal{D}$ is quasilinear rationalizable by $U(x) + y$. □

Lemma A.7. *Suppose $\mathcal{D}$ is quasilinear rationalizable and $\mathcal{D} = \Gamma^Q(\mathcal{D})$. Then $x^i \approx^Q x^j$ if, and only if, $(p^i \cdot x^j - 1) + (p^j \cdot x^i - 1) = 0$.*

Proof. Sufficiency follows by definition of $\approx^Q$. For necessity suppose $\mathcal{D}$ is cyclically monotone and $x^i \approx^Q x^j$. By definition of $\mathcal{D}^Q_\wedge$ we have both $p^i \leq p^j$ and $p^j \leq p^i$, therefore $p^i = p^j$. Since $p^i \cdot x^i = p^j \cdot x^j = 1$, we have the desired result. □

Lemma A.8. *Let $\mathcal{Z}(i)$ be the set of all finite sequences of observations $(m_\ell)_{\ell\in[L]}$ satisfying $x^{m_L} = x^i$. Define*

$$v^i = \min_{\mathcal{Z}(i)} (p^{m_1} \cdot x^{m_2} - 1) + (p^{m_2} \cdot x^{m_3} - 1) + \ldots + (p^{m_{L-1}} \cdot x^i - 1)\,.$$

If $\mathcal{D}$ is quasilinear rationalizable and $\mathcal{D} = \Gamma^Q(\mathcal{D})$ then $v^i \geq v^j + 1 - p^i \cdot x^j$ for every $i, j \in [N]$, $v^i = v^j + 1 - p^i \cdot x^j$ whenever $x^i \approx^Q x^j$, and there is $i' \in [N]$ such that $v^m > v^j + 1 - p^m \cdot x^j$ whenever $x^{i'} \approx^Q x^j$ and $x^{i'} \not\approx^Q x^m$.

Proof. That v^i are well defined, is assured by cyclical monotonicity of $\mathcal{D}$, as whenever an observation is repeated in the sequence, removing the cycle cannot increase the value. Hence there is a minimizer that has no cycles, and since the number of sequences with no cycles is finite, the minimum exists. Let $(x^{m_\ell})_{\ell\in[L]} \in \mathcal{Z}(j)$ such that $v^j = (p^{m_1}\cdot x^{m_2}-1)+(p^{m_2}\cdot x^{m_3}-1)+\ldots+(p^{m_{L-1}}\cdot x^j-1)$. Then by definition of v^i we have

$$\begin{aligned}
v^j + 1 - p^i\cdot x^j &= (p^{m_1}\cdot x^{m_2}-1)+(p^{m_2}\cdot x^{m_3}-1)+\ldots+(p^{m_{L-1}}\cdot x^j-1)+(1-p^i\cdot x^j)\\
&\geq v^i\,.
\end{aligned}$$

Take $x^i \approx^Q x^j$. By Lemma A.7 we have $1-p^i\cdot x^j = -(1-p^j\cdot x^i)$. As $v^i \geq v^j+1-p^i\cdot x^j$ and $v^j \geq v^i + 1 - p^j\cdot x^i$ we have

$$\begin{aligned}
v^i &\geq v^j + (1-p^i\cdot x^j)\\
&= v^j - (1-p^j\cdot x^i)\\
&\geq v^i\,.
\end{aligned}$$

Hence all the inequalities have to be equalities, and in particular $v^i = v^j + (1-p^i\cdot x^j)$.

Finally, towards a contradiction suppose for every $i\in[N]$ there are $j,m\in[N]$ such that $x^i \approx^Q x^j$, $x^i \not\approx x^m$, and $v^m = v^j + 1 - p^m\cdot x^j$. Since $x^i\approx^Q x^j$ we have

$$v^m = v^j + 1 - p^m\cdot x^j = v^i + 1 - p^i\cdot x^j + 1 - p^m\cdot x^j$$

Thus we can construct an infinite sequence $(x^{m_\ell})_{\ell=1}^{\infty}$ such that for all ℓ odd we have $x^{m_\ell}\approx^Q x^{m_{\ell+1}}$ and $x^{m_\ell}\not\approx^Q x^{m_{\ell+2}}$. Furthemore

$$\begin{aligned}
v^{m_1} &= v^{m_2} + (1-p^{m_1}\cdot x^{m_2})\\
&= v^{m_3} + (1-p^{m_2}\cdot x^{m_3}) + (1-p^{m_1}\cdot x^{m_2})\\
&\;\vdots\\
&= v^{m_\ell} + \sum_{r=1}^{\ell-1}(1-p^{m_r}\cdot x^{m_{r+1}})
\end{aligned}$$

$$\vdots$$

As the sequence is infinite, there are $\ell', \ell'' \in \mathbb{N}$ such that $\ell' + 2 < \ell''$ and $m_{\ell'} = m_{\ell''}$. We have

$$\begin{aligned} v^{m_{\ell'}} + \sum_{\ell=1}^{\ell'-1}(1 - q^{m_\ell, Q} \cdot x^{m_{r+1}}) &= v^{m_{\ell''}} + \sum_{\ell=1}^{\ell''-1}(1 - q^{m_\ell, Q} \cdot x^{m_{\ell+1}}) \\ &= v^{m_{\ell''}} + \sum_{\ell=1}^{\ell'-1}(1 - q^{m_\ell, Q} \cdot x^{m_{\ell+1}}) + \sum_{\ell=\ell'}^{\ell''-1}(1 - q^{m_\ell, Q} \cdot x^{m_{\ell+1}}) \end{aligned}$$

Since $v^{m_{\ell'}} = v^{m_{\ell''}}$ and $x^{m_{\ell'}} = x^{m_{\ell''}}$

$$0 = (1 - q^{m_{\ell''-1}, Q} \cdot x^{m_{\ell'}}) + \sum_{\ell=\ell'}^{\ell''-2}(1 - q^{m_\ell, Q} \cdot x^{m_{\ell+1}})$$

Hence $x^{m_{\ell'}} \approx^Q x^{m_{\ell'+2}}$, a contradiction. □

Lemma A.9. *If $\mathcal{D}$ is cyclically monotone and $\mathcal{D} = \Gamma(\mathcal{D})$, then there are numbers $u^i \in \mathbb{R}$ such that $u^i > u^j + 1 - p^i \cdot x^j$ whenever $x^i \not\approx^Q x^j$, and $u^i = u^j + 1 - p^i \cdot x^j$ whenever $x^i \approx^Q x^j$.*

Proof. We proceed by induction on the numbers of observations. If $N = 1$ then $u^1 = 1$ satisfies the conditions.

Suppose the conditions hold for any data set comprised of $N - 1$ or less observations, and take $\mathcal{D}$ comprised of N observations. Take the numbers v^i defined in Lemma A.8 and $i' \in [N]$ such that $v^m > v^i + 1 - p^m \cdot x^i$ whenever $x^{i'} \approx^Q x^i$ and $x^{i'} \not\approx^Q x^m$. Denote $E = \{i \in [N] : x^i \approx^Q x^{i'}\}$ and $D = [N] \setminus E$. If $D = \emptyset$ then the condition is assured by Lemma A.8.

If $D \neq \emptyset$, then $(p^i, x^i)_{i \in D}$ is a data set comprised of $N - 1$ or less observations. By induction hypothesis there are numbers u^m for $m \in D$ such that

$$u^m = u^{m'} + 1 - p^m \cdot x^{m'} \qquad \text{for all } m, m' \in D\,.$$

Take the numbers v^i defined in Lemma A.8 and define

$$\begin{aligned}
\alpha &= \min_{\substack{i\in E\\ m\in D}} v^m - v^j + p^m \cdot x^j - 1 \\
u^i &= v^i + \frac{\alpha}{2} && \text{for all } i \in E\,.
\end{aligned}$$

Hence

$$\begin{aligned}
u^i &> v^m + 1 - p^i \cdot x^m && \text{for all } i \in E \text{ and } m \in D \\
v^m &> u^i + 1 - p^m \cdot x^i && \text{for all } i \in E \text{ and } m \in D \\
u^i &= u^j + p^i \cdot x^j && \text{for all } i, j \in E\,.
\end{aligned}$$

Take a sequence $\beta^n \to 1$, where $\beta^n \in (0,1)$ for all n, and for every $m \in D$ define

$$w^m(n) = \beta^n\, v^m + (1-\beta^n)\tilde{u}^m\,.$$

Take any $m, m' \in D$ and $n \in \mathbb{N}$. If $x^m \approx^Q x^{m'}$ then

$$\begin{aligned}
w^m(n) &= \beta^n\, v^m + (1-\beta^n)u^m \\
&= \beta^n(v^{m'} + 1 - p^m \cdot x^{m'}) + (1-\beta^n)(u^{m'} + 1 - p^m \cdot x^{m'}) \\
&= w^{m'}(n) + 1 - p^m \cdot x^{m'}
\end{aligned}$$

and if $x^m \not\approx^Q x^{m'}$ then

$$\begin{aligned}
w^m(n) &= \beta^n\, v^m + (1-\beta^n)u^m \\
&> \beta^n(v^{m'} + 1 - p^m \cdot x^{m'}) + (1-\beta^n)(\tilde{u}^{m'} + 1 - p^m \cdot x^{m'}) \\
&= w^{m'}(n) + 1 - p^m \cdot x^{m'}
\end{aligned}$$

Since $w^m(n) \to v^m$ for every $m \in D$ for n_0 large enough we have

$$\begin{aligned} & u^i > w^m(n_0) + 1 - p^i \cdot x^m \\ & w^m(n_0) > u^i + 1 - p^m \cdot x^i \end{aligned} \qquad \text{for all } i \in E \text{ and } m \in D\,.$$

Setting $u^m = w^m(n_0)$ for every $m \in D$ assures that the numbers u^i satisfy the desired properties. □

Proof of Proposition A.5. Suffieincy is immediate. For necessity take $\mathcal{D}$ being quasi-linear rationalizable and satisfying $\mathcal{D} = \Gamma^Q(\mathcal{D})$. Take the numbers $u^i \in \mathbb{R}$ from Lemma A.9 and define the functions $\phi^i(x) = u^i + p^i \cdot x - 1$ and $V(x) = \min_{i \in [N]} \phi^i(x)$. Then each ϕ^i is continuous, concave, and strictly increasing, therefore V also is. Moreover, if $x^i \approx^Q x^j$ then $\phi^i(x) = \phi^j(x)$ for all x, and there is $\eta > 0$ such that $V(x^i - \xi) = \phi^i(x^i - \xi)$ whenever $\xi \in B(\eta)$. Define $\tilde{U}(x) = (V \star \rho_\eta)(x)$, where ρ_η is the function defined in the Proof of Proposition 2 (Appendix C), but using the value of η defined here. Then $\tilde{U}$ is continuous, infinitely differentiable, strictly concave and strictly increasing. Moreover

$$\begin{aligned} \tilde{U}(x^i) &= \int_{B(\eta)} \left(u^i + p^i \cdot (x^i - \xi) - 1\right) \rho_\eta(\xi) d\xi \\ &= u^i \int_{B(\eta)} \rho_\eta(\xi) d\xi - p^i \cdot \int_{B(\eta)} \xi \rho_\eta(\xi) d\xi \\ &= u^i \end{aligned} \tag{13}$$

The second equality splits terms and replaces $p^i \cdot x^i = 1$, and the third one follows from $\int_{B(\eta)} \rho_\eta(\xi) d\xi = 1$ and $\int_{B(\eta)} \xi \rho_\eta(\xi) d\xi = \mathbf{0}$.

Let U be the restriction of $\tilde{U}$ to $\mathbb{R}^K_+$. For every x we have

$$\begin{aligned} U(x) &= \int_{B(\eta)} V(x - \xi) \rho_\eta(\xi)\, d\xi \\ &\leq \int_{B(\eta)} \phi^i(x - \xi) \rho_\eta(\xi)\, d\xi \end{aligned}$$

$$
\begin{aligned}
&= [u^i + p^i \cdot x - 1] \int_{B(\eta)} \rho_\eta(\xi)\, d\xi - p^i \cdot \int_{B(\eta)} \xi \rho_\eta(\xi)\, d\xi \\
&= u^i + p^i \cdot x - 1 \\
&= U(x^i) - p^i \cdot x^i + p^i \cdot x\,.
\end{aligned}
$$

The second line follows from $i \in [N]$; the third one splits terms; the fourth one from $\int_{B(\eta)} \rho_\eta(\xi) d\xi = 1$ and $\int_{B(\eta)} \xi \rho_\eta(\xi) d\xi = \mathbf{0}$; and the fifth one from $p^i \cot x^i = 1$ and (13). We conclude that $\mathcal{D}$ is quasilinear rationalizable by $U(x)+y$ and U is infinitely differentiable. $\square$

Proof of Theorem A.3. That Q-1) implies Q-2) follows from an iterative application of Proposition A.4. That Q-2) implies Q-3) follows from taking the numbers u^i from Lemma A.9 applied to $\mathcal{D}^Q_\wedge = (q^{i,Q}, x^i)_{i\in[N]}$, and defining $\mu^i = p^i - q^{i,Q}$. Starting from Q-3) we can construct an infinitely differentiable U such that $U(x) + y$ quasilinear rationalizes $\mathcal{D}$ by a construction similar to the one in the proof of Proposition A.5; the only difference is that in this case $\phi^i = u^i + (p^i - \mu^i) \cdot x - y$. Finally, that Q-4) implies Q-1) is immediate. $\square$

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