Preference Recoverability from Inconsistent Choices
Online Appendix

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1 The Varian Index

1.1 Definition

The Varian Index (Varian, 1990) follows the idea of v-rationalization (Definition 7 in the main body of the paper).[1] Starting from a vector $v \in [0,1]^N$ we define v-revealed preferences as follows:

Definition 1. Given vector $v \in [0,1]^N$, and two choices x^i, x^j,

- x^i is *v-directly revealed preferred* to x^j, denoted $x^i \succsim_v^D x^j$ if (1) $x^i = x^j$, or (2) $p^i x^j \leq v_i$;
- x^i is *v-directly revealed strictly preferred* to x^j, denoted $x^i \succ_v^D x^j$ if $p^i x^j < v_i$;
- x^i is *v-revealed preferred* to x^j, denoted $x^i \succsim_v^R x^j$ if there is a sequence of choices $(x^{m_\ell})_{\ell \in [L]}$ such that $x^i \succsim_v^D x^{m_1} \succsim_v^D \dots \succsim_v^D x^{m_L} \succsim_v^D x^j$; and
- x^i is *v-revealed strictly preferred* to x^j, denoted $x^i \succ_v^R x^j$, if there are choices x^m, $x^{m'}$ such that $x^i \succsim_v^R x^m \succ_v^D x^{m'} \succsim_v^R x^j$.

Similarly, we can define a relaxed version of GARP

Definition 2. For a vector $v \in [0,1]^N$ the choice data $\mathcal{D}$ satisfies the *Generalized Axiom of Revealed Preferences given v* (GARP$_v$) if for every pair of choices x^i, x^j, if $x^i \succsim_v^R x^j$ then $x^j \not\succ_v^D x^i$.

Afriat (1973) shows that when $v = e\mathbf{1}$ for $e \in [0,1]$, GARP$_v$ is equivalent to the data being v-rationalizable, and (as his original theorem) the preference relation v-rationalizing

[1] This section follows the exposition presented by Halevy et al. (2018).

the data can always be chosen to be continuous and convex. The general equivalence between GARP$_v$ and v-rationalization (for any vector $v \in [0,1]^N$) is shown in Halevy et al. (2018).[2] As with the result in Afriat (1973), in this general case, it is also without loss of generality to choose a continuous and convex preference relation rationalizing the data.

Theorem (Halevy et al. (2018), Theorem 1)**.** *A choice data is v-rationalizable by a continuous and monotone preference relation if, and only if, it satisfies GARP$_v$.*

When $v = \mathbf{1}$, the previous result is equivalent to the original version of the Afriat Theorem. Varian (1990) proposes to use as vector v the one that minimizes the distance with $\mathbf{1}$ (in some metric) among the ones that satisfy GARP. In general, the measure is defined as follows:

Definition 3. Let $f : [0,1]^N \to [0,1]$ be a continuous and weakly decreasing function satisfying $f(\mathbf{0}) = 1$ and $f(\mathbf{1}) = 0$. The *Varian efficiency Index* is

$$I_V(\mathcal{D}) = \inf_{\{v \in [0,1]^N : \mathcal{D} \text{ satisfies GARP}_v\}} f(v) \,. \tag{1}$$

Take a vector $v \in [0,1]^N$ such that $\mathcal{D}$ satisfies GARP$_v$. We know that a preference that v-rationalizes the data will agree with all the v-revealed preferences. Starting from this point, we can easily characterize the out-of-sample accuracy test. For this, we define the tuple of mistakes as $\mathcal{M} = \left(\succsim^D \setminus \succsim^D_v, \succ^D \setminus \succ^D_v\right)$, use the vector v as the $v^{\mathcal{M}}$ vector, and apply Proposition 6 in the main body of the paper.

1.2 Difference between the MM and the Varian estimators

Figure 1 shows choice data with three different observations. The directly revealed strict preference relations are

$$x^1 \succ^D x^2,\, x^2 \succ^D x^1,\, x^2 \succ^D x^3, \text{and } x^3 \succ^D x^1.$$

The set of preferences recovered using the Varian Index and the MM Index may differ: The Varian Index (as shown in the picture) interprets as "correct" the revealed preferences $x^1 \succ^D x^2$ and $x^3 \succ^D x^1$; hence any recovered preference relation $\succsim_V$ will satisfy $x^3 \succ_V x^1 \succ_V x^2$. On the other hand, the solution to the MM Index interprets as "correct" the

[2] Although the equivalence between GARP$_v$ and v-rationalizability appears to be the primary motivation in Varian (1990), neither he nor any further work following it proved such a result before Halevy et al. (2018).

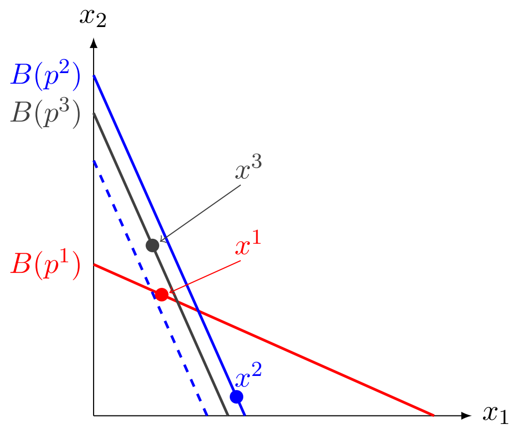


Figure 1: Example of choice data where the preferences recovered using the MM and Varian indices differ. Dotted blue line shows how the Varian Index shrinks the budget set of the second observation.

preferences $x^2 \succ^D x^1$, $x^2 \succ^D x^3$, and $x^3 \succ^D x^1$; hence any recovered preference $\succsim_M$ will satisfy $x^2 \succ_M x^3 \succ_M x^1$.

1.3 Computation Details

We compute the Varian Index using the mixed-integer linear programming approach developed by Demuynck and Rehbeck (2023). For the objective function $f(v)$, we use the (normalized) distance between the vector v and a vector of ones in the taxicab geometry:

$$f(v) = \frac{1}{N} \sum_{i \in [N]} (1 - v_i) \tag{2}$$

We choose the objective function (2) over the most popular normalized Euclidean distance between v and a vector of ones for practical reasons. Although both problems are known to be NP-hard, in practice, we find that the computation of the Varian Index is superior with a linear objective than with a quadratic one, both in terms of computation time and success rate. Table 1 presents summary statistics of computations under both objective functions when the time limit to compute the index for each subject is set to one hour. Since the subjects for whom the Varian Index cannot be computed with Euclidean distance are likely to present more violations of GARP, using this objective would bias our estimation, as we would only include a non-random portion of our sample. Hence, we use the Varian index using the taxicab distance to include all the subjects in the sample.

Table 1: Varian index - Summary Statistics

Sample		# Sub	Objective	success rate	Time average	Time 95$^{\text{th}}$ percentile
2D	Risk - General	951	linear	100%	6.11	6.81
			quadratic	100%	15.37	24.05
	Risk - Students	762	linear	100%	24.92	30.08
			quadratic	88.32%	466.00	1323.51
	Social - General	1549	linear	100%	27.86	32.89
			quadratic	79.72%	549.36	1754.91
	Social - Students	709	linear	100%	25.20	28.75
			quadratic	90.46%	450.69	1349.14
	ALL	3971	linear	100%	21.61	31.06
			quadratic	88.14%	388.09	1360.12
3D	Risk	141	linear	100%	24.35	27.62
			quadratic	85.82%	570.45	2268.58
	Ambiguity	134	linear	100%	31.26	35.27
			quadratic	90.30%	539.32	1994.74
	Social	45	linear	100%	25.34	29.23
			quadratic	82.22%	225.88	907.23
	ALL	320	linear	100%	27.38	33.87
			quadratic	87.19%	508.96	2014.50

Summary Statistics for Varian Index succes rate and computation time under different objective functions. D is number of dimensions, # Sub is number of subjects who fail GARP. Success rate is computed over number of subjects who fail GARP. Time limit to compute the index for each subject is set to one hour.

2 Comparison of the MM and Varian Estimators

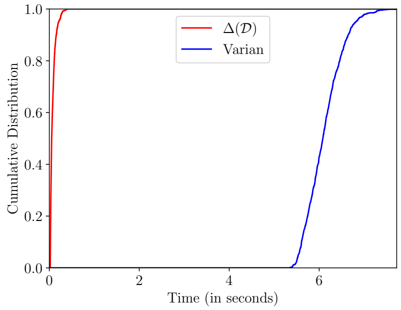
2.1 Computational complexity and computation time

Computing both $\Delta(\mathcal{D})$ and the Varian Index are NP-Hard problems. To see that $\Delta(\mathcal{D})$ is NP-hard, note that by Proposition 2 its computation reduces to solving a Minimum Feedback Arc Set problem, which is one of the 21 original NP-complete problems in Karp (1972). Smeulders et al. (2014) show that computing the Varian Index is NP-Hard. We solve the MFAS problem necessary to compute the $\Delta(\mathcal{D})$ and the MM estimator using the methodology developed by Baharev et al. (2021), which implementation is available in Baharev (2021). The Varian index is computed using the method developed by Demuynck and Rehbeck (2023).

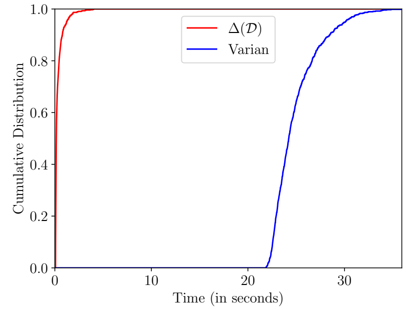
Figure 2 shows the cumulative distribution of computation time for both $\Delta(\mathcal{D})$ and the Varian Index. For all sub-samples, the computation of $\Delta(\mathcal{D})$ is significantly faster than the one of the Varian Index, both in average and for the subjects who present a higher index level (i.e., a lower level of rationality).

2.2 Ordering and out-of-sample accuracy

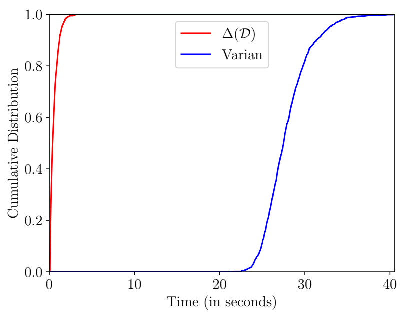
Figure 3 shows the ordering of subjects according to $\Delta(\mathcal{D})$ and the Varian Index per sub-sample, and Table 2 presents the out-of-sample accuracy and completeness for both indices using five choices to test the data instead of ten. In both cases, the results are similar to the ones presented in the main body of the paper.



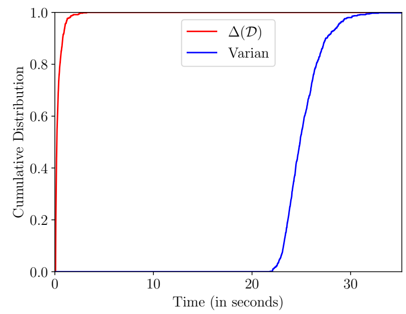
(a) 2D - Risk - General



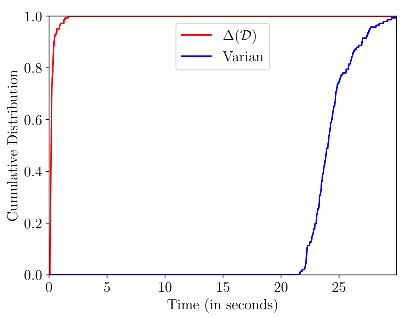
(b) 2D - Risk - Students



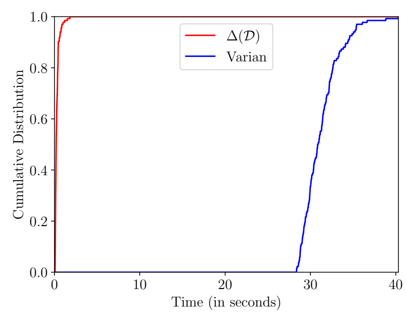
(c) 2D - Social - General



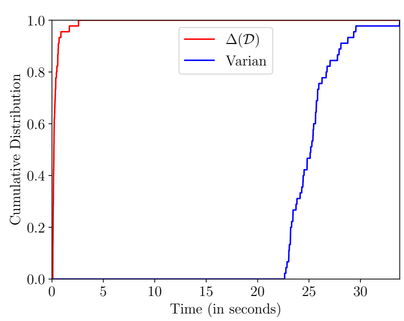
(d) 2D - Social - Students



(e) 3D - Risk

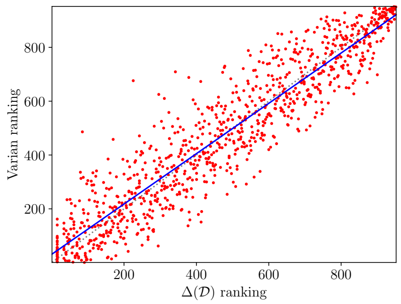


(f) 3D - Ambiguity

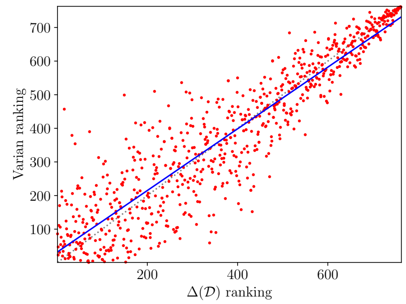


(g) 3D - Social

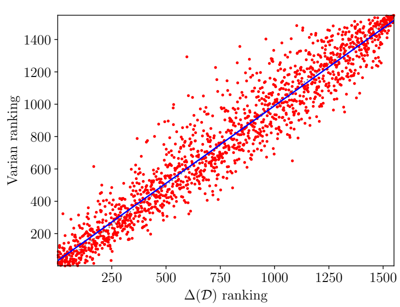
Figure 2: Cumulative distribution of computation time for $M(\mathcal{D})$ and Varian Index.



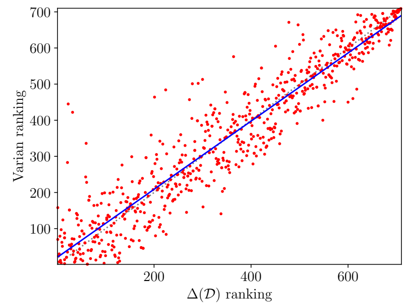
(a) 2D - Risk - General



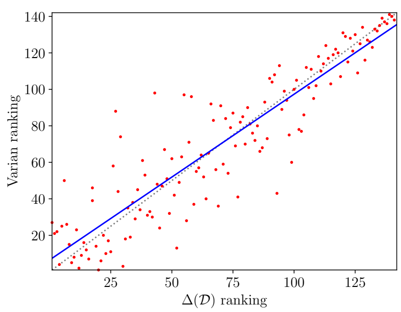
(b) 2D - Risk - Students



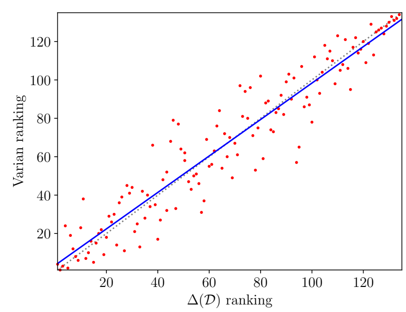
(c) 2D - Social - General



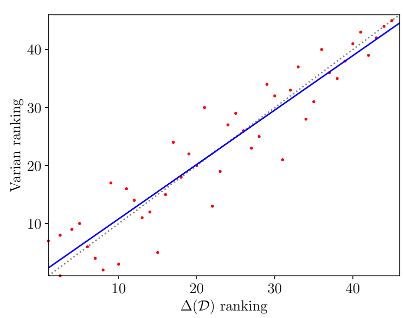
(d) 2D - Social - Students



(e) 3D - Risk



(f) 3D - Ambiguity



(g) 3D - Social

Figure 3: Ordering of subjects according to $M(\mathcal{D})$ and Varian Index. Linear regression in blue; gray dotted line is 45° line.

Table 2: Out-of-sample prediction (Test: 5 choices)

Sample		# Sub	Accuracy MM	Accuracy Varian	Accuracy Diff	Completeness MM	Completeness Varian	Completeness Diff
2D	Risk - General	951	0.721	0.725	-0.004	0.316	0.336	-0.020
			(0.007)	(0.007)	(0.003)	(0.018)	(0.017)	(0.007)
	Risk - Students	762	0.805	0.776	0.029	0.618	0.604	0.014
			(0.008)	(0.009)	(0.004)	(0.014)	(0.014)	(0.006)
	Social - General	1549	0.689	0.690	-0.001	0.422	0.430	-0.007
			(0.006)	(0.006)	(0.003)	(0.012)	(0.011)	(0.005)
	Social - Students	709	0.768	0.780	-0.011	0.565	0.597	-0.032
			(0.008)	(0.008)	(0.003)	(0.017)	(0.015)	(0.007)
	ALL	3971	0.733	0.731	0.002	0.460	0.470	-0.011
			(0.004)	(0.004)	(0.002)	(0.008)	(0.007)	(0.003)
3D	Risk	141	0.794	0.733	0.061	0.581	0.529	0.052
			(0.018)	(0.022)	(0.010)	(0.036)	(0.035)	(0.015)
	Ambiguity	134	0.797	0.737	0.060	0.618	0.551	0.067
			(0.017)	(0.020)	(0.012)	(0.029)	(0.030)	(0.019)
	Social	45	0.640	0.569	0.071	0.403	0.375	0.029
			(0.034)	(0.041)	(0.021)	(0.055)	(0.052)	(0.025)
	ALL	320	0.774	0.712	0.062	0.572	0.517	0.055
			(0.012)	(0.014)	(0.007)	(0.022)	(0.021)	(0.011)

Out-of-sample average accuracy and completeness for each sub-sample, including only subjects who fail GARP. standard errors in parenthesis. Diff is difference between MM Varian estimators, D is number of dimensions, and # Sub. is number of subjects. Test data is 5 observations, and the remaining are used for training.

3 Additional Proofs

3.1 Remarks

Proof of Remark 1. For necessity suppose $x^i \succ^D x^j$, and take y satisfying $g^i(y) = 1$ and $y \rhd x^j$. Since $\unrhd$ is continuous, for ε small enough we have $y - \varepsilon\mathbf{1} \rhd x^j$, which implies $y - \varepsilon\mathbf{1} \unrhd x^j$, and $g^i(y - \varepsilon\mathbf{1}) < 1$ holds as g^i is increasing.

For sufficiency suppose there is y such that $g^i(y) < 1$ and $y \unrhd x^j$. Let $\alpha = \max_{k\in[K]} x^i_k - y_k$. Then $y + 2\alpha\mathbf{1} > x^i$, which as g^i is increasing implies $g^i(y+2\alpha\mathbf{1}) > 1$. As g^i is continuous and $g^i(y) < 1$ there is $\beta \in (0, 2\alpha)$ such that $g^i(y + \beta\mathbf{1}) = 1$, and as $\unrhd$ extends $\geq$, $y \unrhd x^j$ implies $y + \beta\mathbf{1} \rhd x^j$. Therefore $x^i \succ^D x^j$. $\square$

Proof of Remark 2. As $\mathcal{D}$ satisfies GARP$_{\mathcal{M}}$ by Theorem 3 there is a continuous and monotone preference relation $\succsim \in \mathcal{R}(\mathcal{M})$. As $\succsim$ is continuous by Debreu (1954) Theorem it has a continuous utility representation $u : \mathbb{R}^K_+ \to \mathbb{R}$. As u is continuous and $\{x : g(x) \leq 1\}$ is compact, by Weierstrass Theorem u attains a maximum value $x^\star$ on the budget set generated by g. Therefore $x^\star \in C_{\mathcal{M}}(g)$. $\square$

3.2 Proof of Proposition 6

Proof of Proposition 6. To show necessity suppose $x \in C_{\mathcal{M}}(g)$. This implies that there is a continuous and $\unrhd$-monotone preference relation $\succsim$ that $v^{\mathcal{M}}$ rationalizes the data and discards only $\mathcal{M}$, and for which $x \succsim y$ whenever $g(y) \leq 1$. Towards a contradiction suppose either (6), (7), or (8) fails.

- If (6) x is clearly not optimal as there is $y \in B$ such that $y \succ x$ (as $\succsim$ is $\unrhd$-monotone).
- If (7) fails then there is $i \in [N]$ such that $g \gtrsim^R_{\mathcal{M}} g^i$ and either $x^i \rhd x$ or there is y such that $g^i(y) = v_i$ and $y \rhd x$. As $g \gtrsim^R_{\mathcal{M}} g^i$ there is a chosen bundle x^j such that $g \gtrsim^D_{\mathcal{M}} g^j$ and $x^i \succsim^R_{\mathcal{M}} x^j$. Since $g \gtrsim^D_{\mathcal{M}} g^j$ there is y such that $g(y) = 1$ and $y \unrhd x^j$, which as $x \in C_{\mathcal{M}}(g)$ implies that there is an $\unrhd$-monotonic preference relation $\succsim \in \mathcal{R}(\mathcal{M})$ for which $x \succsim y$, so $\unrhd$-monotonocity and transitivity imply $x \succsim x^j$. As $\succsim$ discards only $\mathcal{M}$ transitivity implies $x^j \succsim x^i$, therefore $x \succsim x^i$. There are two possible cases: (i) $x^i \rhd x$, and (ii) there is y such that $g^i(y) = v^{\mathcal{M}}_i$ and $y \rhd x$. In (i), $\unrhd$-monotonicity implies $x^i \succ x$. In (ii), as $\succsim$ $v^{\mathcal{M}}$ rationalizes the data we have $x^i \succsim y$, and $\unrhd$-monotonicity implies $y \succ x$. In both cases we found a contradiction with $x \succsim x^i$.
- If (8) fails then there is $i \in [N]$ such that $g \gtrdot^R_{\mathcal{M}} g^i$ and $x^i \rhd x$ or there is y such that $g^i(y) = 1$ and $y \rhd x$. As $g \gtrdot^R_{\mathcal{M}} g^i$ we have one of the two following cases: (i) there is a bundle y and a chosen bundle x^j such that $g(y) = 1$, $y \rhd x^j$, and $x^j \succsim^R_{\mathcal{M}} x^i$, or (ii)

there is a bundle y and a chosen bundle x^j such that $g(y) = 1$, $y \trianglerighteq x^j$, and $x^j \succ^R_{\mathcal{M}} x^i$. In (i), as $x \in C_{\mathcal{M}}(g)$ there is a preference relation $\succsim \in \mathcal{R}(\mathcal{M})$ such that $x \succsim y$. Then $\trianglerighteq$-monotonicity implies $y \succ x^j$, and by transitivity $x \succ x^j$. As $\succsim$ discards only $\mathcal{M}$ we have that $x^j \succsim x^i$, thus $x \succ x^i$. In (ii), as $x \in C_{\mathcal{M}}(g)$ there is a preference relation $\succsim \in \mathcal{R}(\mathcal{M})$ such that $x \succsim y$. Then $\trianglerighteq$-monotonicity implies $y \succsim x^j$, and by transitivity $x \succsim x^j$. As $\succsim$ discards only $\mathcal{M}$ we have that $x^j \succ x^i$, thus $x \succ x^i$. In both cases we conclude $x \succ x^i$.
As (8) fails one of the following holds. (a) $x^i \trianglerighteq x$, or (b) there is y such that $g^i(y) = v_i^{\mathcal{M}}$ and $y \trianglerighteq x$. In (a) $\trianglerighteq$-monotonicity implies $x^i \succsim x$. In (b), as $\succsim$ $v^{\mathcal{M}}$ rationalizes the data we have $x^i \succsim y$, and $\trianglerighteq$-monotonicity implies $y \succsim x$. From transitivity we have $x^i \succsim x$. In both cases we found a contradiction with $x \succ x^i$.

For sufficiency suppose equations (6), (7), and (8) hold. Let $g^{N+1} = g$, $x^{N+1} = x$, and define the extended data $\mathcal{D}_e = (p^i, x^i)_{i \in [N+1]}$ ((6) assures $g^{N+1}(x^{N+1}) = 1$). Also define the extended couple of mistakes $\mathcal{M}_e = (\mathcal{M}_e^w, \mathcal{M}_e^s)$ by

$$\mathcal{M}_e^w = \mathcal{M}^w \bigcup \{(x^i, x^{N+1}) \in \succsim^D : i \leq N,\ x^i \ntrianglerighteq x^{N+1}, \text{ and } \nexists\ y \text{ s.t. } g^i(y) = v_i^{\mathcal{M}} \text{ and } y \triangleright x^{N+1}\}$$
$$\mathcal{M}_e^s = \mathcal{M}^s \bigcup \{(x^i, x^{N+1}) \in \succ^D : i \leq N,\ x^i \ntriangleright x^{N+1}, \text{ and } \nexists\ y \text{ s.t. } g^i(y) = v_i^{\mathcal{M}} \text{ and } y \trianglerighteq x^{N+1}\}$$

Then $\mathcal{M}_e^w$ and $\mathcal{M}_e^s$ have mistakes of the form (x^i, x^j) with $i, j \in [N]$ if and only if (x^i, x^j) are also in $\mathcal{M}^w$ and $\mathcal{M}^s$, respectively. Furthermore, we have $x^i \succsim^D_{\mathcal{M}_e} x^{N+1}$ if and only if either $x^i \trianglerighteq x^{N+1}$ or there is y such that $g^i(y) = v_i^{\mathcal{M}}$ and $y \trianglerighteq x^{N+1}$; and $x^i \succ^D_{\mathcal{M}_e} x^{N+1}$ if and only if either $x^i \triangleright x^{N+1}$ or there is y such that $g^i(y) = v_i^{\mathcal{M}}$ and $y \triangleright x^{N+1}$.[3]

If $\mathcal{D}_e$ satisfies GARP$_{\mathcal{M}_e}$ then for any $\mathcal{M}_e$ vector $v^{\mathcal{M}_e}$ there is a preference relation that v^{M_e}-rationalizes the data and discards only $\mathcal{M}_e$. In particular this is true for $v^{\mathcal{M}_e}$ defined as

$$v_i^{\mathcal{M}_e} = \begin{cases} v_i^{\mathcal{M}} & \text{if } i \in [N] \\ 1 & \text{if } i = N+1\,. \end{cases}$$

Given how $\mathcal{M}_e$ is constructed, if $\succsim$ $v^{\mathcal{M}_e}$-rationalizes $\mathcal{D}_e$ and discards only $\mathcal{M}_e$, then it

[3]For sufficiency in the case of $\succsim^D_{\mathcal{M}^e}$ suppose either $x^i \trianglerighteq x^{N+1}$ or there is y such that $g^i(y) = v_i^{\mathcal{M}}$ and $y \trianglerighteq x^{N+1}$. If $x^i \trianglerighteq x^{N+1}$ then $x^i \succsim^D x^{N+1}$ and $(x^i, x^{N+1}) \notin \mathcal{M}_e^w$, therefore $x^i \succsim^D_{\mathcal{M}_e} x^{N+1}$. If there is y such that $g^i(y) = v_i^{\mathcal{M}}$ and $y \trianglerighteq x^{N+1}$, then there is $\varepsilon \geq 0$ for which $y' = y + \varepsilon \mathbf{1}$ is such that $g^i(y') = 1$. As $\trianglerighteq$ extends $\geq$ we have $y' \trianglerighteq y$, which by transitivity implies $y' \trianglerighteq x^{N+1}$, so $x^i \succsim^D x^{N+1}$. As $(x^i, x^{N+1}) \notin \mathcal{M}_e^w$ we have $x^i \succsim^D_{\mathcal{M}_e} x^{N+1}$.

For necessity in the case of $\succsim^D_{\mathcal{M}^e}$ suppose $x^i \succsim^D_{\mathcal{M}_e} x^{N+1}$. This is, $x^i \succsim^D x^{N+1}$ and $(x^i, x^{N+1}) \notin \mathcal{M}_e^w$. As $x^i \succsim^D x^{N+1}$ either $x^i \trianglerighteq x^{N+1}$ or there is y' such that $g^i(y') = 1$ and $y' \trianglerighteq x^{N+1}$. If $x^i \trianglerighteq x^{N+1}$ then the condition holds. If there is y' such that $g^i(y') = 1$ and $y' \trianglerighteq x^{N+1}$, towards a contradiction suppose there is no y such that $g^i(y) = v_i^{\mathcal{M}}$ and $y \trianglerighteq x^{N+1}$. Then as $x^i \ntrianglerighteq x^{N+1}$ we have that $(x^i, x^{N+1}) \in \mathcal{M}_e^w$, which contradicts $x^i \succsim^D_{\mathcal{M}_e} x^{N+1}$.

The case for $\succ^D_{\mathcal{M}_e}$ is analogous.

also $v^{\mathcal{M}}$-rationalizes $\mathcal{D}$ and discards only $\mathcal{M}$, i.e., $\succsim \in \mathcal{R}(\mathcal{M})$. And as $v_{N+1}^{\mathcal{M}_e} = 1$ then $x^{N+1} \succsim y$ whenever $g^{N+1}(y) \leq 1$, i.e., $x^{N+1} \in C_{\mathcal{M}}(p^{N+1})$. Therefore a sufficient condition for $x \in C_{\mathcal{M}}(p)$ is for $\mathcal{D}_e$ to satisfy $\text{GARP}_{\mathcal{M}_e}$.

Towards a contradiction suppose $\mathcal{D}_e$ fails $\text{GARP}_{\mathcal{M}_e}$. Then there is a chain $(x^{m_\ell})_{\ell \in [L]}$ ($L \geq 2$) such that $x^{m_\ell} \succsim^D_{\mathcal{M}_e} x^{m_{\ell+1}}$ for all $\ell \in [L-1]$ and $x^{m_L} \succ^D_{\mathcal{M}_e} x^{m_1}$. As $\mathcal{D}$ satisfies $\text{GARP}_{\mathcal{M}}$ there has to be an index $\ell' \in [L]$ such that $m_{\ell'} = N+1$. Since (6) assures $x^{N+1} \not\succ^D x^{N+1}$ it is without loss of generality to suppose such ℓ' is unique.[4] If $\ell' = 1$ then $p^{N+1} \gtrsim^R_{\mathcal{M}} p^{m_L}$. As $x^{m_L} \succ^D_{\mathcal{M}_e} x^{N+1}$ we have either $x^{m_L} \rhd x^{N+1}$ or there is y such that $g^{m_L}(y) = v^{\mathcal{M}}_{m_L}$ and $y \rhd x^{N+1}$. This contradicts (7). If $\ell' > 1$ then $p^{N+1} \gtrdot p^{m_{\ell'-1}}$. As $x^{m_{\ell'-1}} \succsim^D_{\mathcal{M}_e} x^{N+1}$ we have either $x^{m_{\ell'-1}} \trianglerighteq x^{N+1}$ or there is y such that $g^i(y) = 1$ and $y \trianglerighteq x$. This contradicts (8). We conclude that $\mathcal{D}_e$ satisfies $\text{GARP}_{\mathcal{M}_e}$ and $x \in C_{\mathcal{M}}(p)$. $\square$

[4]If there is $A \subset [L]$ such that $|A| \geq 2$ and $m_{\ell'} = N+1$ for all $\ell' \in A$ we can construct another sequence $(x^{m_s})_{s \in [S]}$ such that (1) $x^{m_s} \succsim^D_{\mathcal{M}_e} x^{m_{s+1}}$ for all $s \in [S-1]$, (2) $x^{m_S} \succ^D_{\mathcal{M}_e} x^{m_1}$, and (3) there is a unique $s' \in [S]$ such that $m_s = N+1$. Define $\underline{\ell} = \min\{\ell : \ell \in A\}$ and $\overline{\ell} = \max\{\ell : \ell \in A \cap [L-1]\}$. If $L \notin A$ remove from the sequence $(x^{m_\ell})_{\ell \in [L]}$ all the elements for which $\ell > \underline{\ell}$ and $\ell \leq \overline{\ell}$. If $L \in A$ remove from the sequence all the elements for which $\ell \leq \overline{\ell}$. The resulting sequence satisfies all the desired properties.

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