The Generality of the Strong Axiom
Online Appendix

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Differentiable Utility under partial efficiency

Here we address the problem of strongly $\mathbf{v}$-rationalizing a data set using a utility function that is continuous, strictly increasing, strictly concave, and infinitely differentiable. We show that under $\mathrm{SARP}_{\mathbf{v}}$, it is possible to recover an infinitely differentiable utility if we impose partial efficiency in all observations, i.e., if $\mathbf{v} \ll \mathbf{1}$. As shown by Chiappori and Rochet (1987), a sufficient condition for differentiability of a strictly concave utility function is for the demand data to be "invertible"; this is, different choices have to come from different prices ($p^i \neq p^j$ implies $x^i \neq x^j$). A non-invertible demand implies that the same bundle is optimal from two different price vectors, which (as income is normalized to one) contradicts the first order condition of equality between the marginal rate of substitution and the price ratio.[1] However, if the partial efficiency level of a choice is less than one, this equality is not necessary anymore. Hence we can find a differentiable utility that $\mathbf{v}$-rationalizes non-invertible choice data.

Theorem 3. *If* $\mathbf{v} \ll \mathbf{1}$ *and* $\mathcal{D}$ *satisfies* $\mathrm{SARP}_{\mathbf{v}}$, *then it is strongly* $\mathbf{v}$*-rationalizable by a strictly increasing, strictly concave, and infinitely differentiable utility function.*

[1] In a strict sense, equality between the marginal rate of substitution and the price ratio is necessary only when goods are consumed in a strictly positive amount. See Ugarte (2023) for more details.

Following the results in Section 3, it is easy to see that a version of Theorem 2 applies to a differentiable utility; this is, if $\mathbf{v} < \mathbf{1}$ and $\mathcal{D}$ satisfies $\text{GARP}_{\mathbf{v}}$, there is $\mathbf{v}^\star \ll \mathbf{1}$ such that $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{v}^\star}$ and $f(\mathbf{v}^\star) = f(\mathbf{v})$.

The proof of Theorem 3 uses a modified version of the Afriat inequalities.

Lemma 7. *If $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{v}}$ and $\mathbf{v} \ll \mathbf{1}$ then there are numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ for $i \in [N]$ such that for all $i, j \in [N]$*

$$u^i > u^j + \lambda^i (v_i - p^i x^j) \tag{1}$$

Proof. By Lemma 2 there are numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ such that (1) holds whenever $x^i \neq x^j$ and $u^i = u^j$ whenever $x^i = x^j$. As $v_i < 1$, whenever $x^i = x^j$ we have $p^i x^j = p^i x^i = 1 > v_i$. Therefore, as $\lambda^i > 0$, $u^i = u^j > u^j + \lambda^i (v_i - p^i x^j)$ whenever $x^i = x^j$. $\square$

Proof of Theorem 3. By Lemma 7 there is $\varepsilon > 0$ small enough such that

$$u^i - \varepsilon g(x^i - x^j) > u^j + \lambda^i (v_i - p^i x^j) \qquad \text{for all } i, j \in [N] \tag{2}$$

$$\lambda^i p^i_k > \varepsilon \qquad \text{for all } i \in [N],\ k \in [K] \tag{3}$$

Define the function

$$U(x) = \min_{i \in [N]} u^i - \lambda^i (v_i - p^i x) - \varepsilon g(x - x^i)\,.$$

Since each function $u^i - \lambda^i (v_i - p^i x) - \varepsilon g(x - x^i)$ is continuous, strictly concave, and strictly increasing,[2] $U(x)$ inherits these properties. Furthermore, from (2) we have

$$U(x^i) = \min_{j \in [J]} u^j - \lambda^j (v_j - p^j x^i) - \varepsilon g(x^j - x^i) > u^i\,.$$

Although U is continuous, strictly increasing, and strictly concave, it is not differentiable. Following Chiappori and Rochet (1987) we smooth U by convolution. For $\delta > 0$ denote by

[2] ϕ^i is strictly increasing since from (3) for all $k \in [K]$ we have

$$\frac{\partial \phi^i(x)}{\partial x_k} = \lambda^i p^i_k - \varepsilon \left(\frac{(x_k)^2}{M + \sum_{k \in [K]} (x_k)^2} \right) > \lambda^i p^i_k - \varepsilon > 0\,.$$

$B(\delta)$ the ball or radius δ centered at $\mathbf{0}$. As $U(x)$ is continuous and $U(x^i) > u^i$, there is η small enough such that for all $i \in [N]$ whenever $x \in B(\eta)$ we have $U(x^i - x) > u^i$. Define

$$\rho_1(x) = \begin{cases} \left[\int_{\mathbb{R}^K} \exp\left(-\frac{1}{||y||^2-1}\right) dy\right]^{-1} \exp\left(-\frac{1}{||x||^2-1}\right) & \text{if } ||x|| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$
$$\rho(x) = \frac{1}{\eta}\rho\left(\frac{x}{\eta}\right)$$

So $\rho(x)$ is continuous, infinitely differentiable,[3] symmetric ($\rho(x) = \rho(-x)$), weakly positive ($\rho(x) \geq 0$), and strictly positive whenever $||x|| < \eta$. It also satisfies

$$\int_{B(\mathbf{0},\eta)} \rho(\xi)\, d\xi = 1, \text{ and} \tag{4}$$
$$\int_{B(\mathbf{0},\eta)} \xi\rho(\xi)\, d\xi = 0\,. \tag{5}$$

Define

$$\begin{aligned} V(x) &= (U \star \rho)(x) \\ &= \int_{\mathbb{R}^K} U(x-\xi)\rho(\xi)\, d\xi \\ &= \int_{B(\eta)} U(x-\xi)\rho(\xi)\, d\xi \end{aligned} \tag{6}$$

V is infinitely differentiable, strictly concave, and strictly increasing (see Chiappori & Rochet, 1987). Finally take x satisfying $p^i\, x \leq v_i$. Then

$$\begin{aligned} V(x) &= \int_{B(\eta)} U(x-\xi)\rho(\xi)\, d\xi \\ &= \int_{B(\eta)} \left[\min_{j\in[N]} u^j - \lambda^j(v_j - p^j\,(x-\xi)) - \varepsilon g(x-\xi-x^j)\right] \rho(\xi)\, d\xi \\ &\leq \int_{B(\eta)} \left[u^i - \lambda^i(v_i - p^i\,(x-\xi)) - \varepsilon g(x-\xi-x^i)\right] \rho(\xi)\, d\xi \\ &= \left[u^i - \lambda^i(v_i - p^i\, x)\right] \int_{B(\eta)} \rho(\xi)\, d\xi - \lambda^i p^i \int_{B(\eta)} \xi\rho(\xi)\, d\xi - \varepsilon \int_{B(\eta)} g(x-\xi-x^i)\rho(\xi)\, d\xi \\ &= u^i - \lambda^i(v_i - p^i x) - \varepsilon \int_{B(\eta)} g(x-\xi-x^i)\rho(\xi)\, d\xi \\ &< u^i - \lambda^i(v_i - p^i x) \end{aligned}$$

[3] In particular all the partial derivatives on $\rho(x)$ of any order equal zero whenever $||x|| = \eta$.

$$
\begin{aligned}
&\leq u^i \\
&= \int_{B(\eta)} u^i \rho(\xi)\, d\xi \\
&< \int_{B(\eta)} U(x^i - \xi)\rho(\xi)\, d\xi \\
&= V(x^i)\,.
\end{aligned}
$$

The second line follows from the definition of U; the third one since $i \in [N]$; the fourth one rearranges terms; the fifth one from (4) and (5); the sixth one from the positivity of g; the seventh one from $\lambda^i > 0$ and $p^i\, x \leq v_i$; the eight one from (4); the ninth one from $u^i < U(x^i - \xi)$ whenever $\xi \in B(\eta)$; and the tenth one from the definition of V. We conclude that V strongly $\mathbf{v}$-rationalizes $\mathcal{D}$. □

References

Chiappori, P.-A., & Rochet, J.-C. (1987). Revealed Preferences and Differentiable Demand. *Econometrica*, *55*(3).

Ugarte, C. (2023). *Smooth rationalization*, Working Paper.