The generality of the Strong Axiom

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Abstract

A widespread assumption in economics is that consumers have strictly convex preferences and hence a (single-valued) demand function. When choices are rationalizable strict convexity can be tested by the Strong Axiom of Revealed Preferences (SARP). We extend this test to non-rationalizable choices by recovering preferences under partial efficiency. Our main result shows that for non-rationalizable choices strict convexity cannot be falsified. Hence a demand function is falsifiable only if choices are rationalizable and fail SARP, which we do not observe in laboratory choice data. Our results suggest that assuming strict convexity does not imply a substantial cost.

1 Introduction

One of the most widespread assumptions in economic research is the strict convexity of agents' preferences, whose main implication is the existence of demand functions instead of correspondences. This assumption simplifies the analysis in both theoretical and empirical research. Most results in general equilibrium, applied game theory, and mechanism and information design rely in this assumption to keep the models tractable. Empirically, demand estimations usually proceed by adding an error term to a parametric demand function which is estimated.

In this paper I study the empirical content of the assumption of strict convexity. I extend the classical analysis to the case when the agent's choices present bounded rationality, i.e., when their choices fail the Generalized Axiom of Revealed preferences

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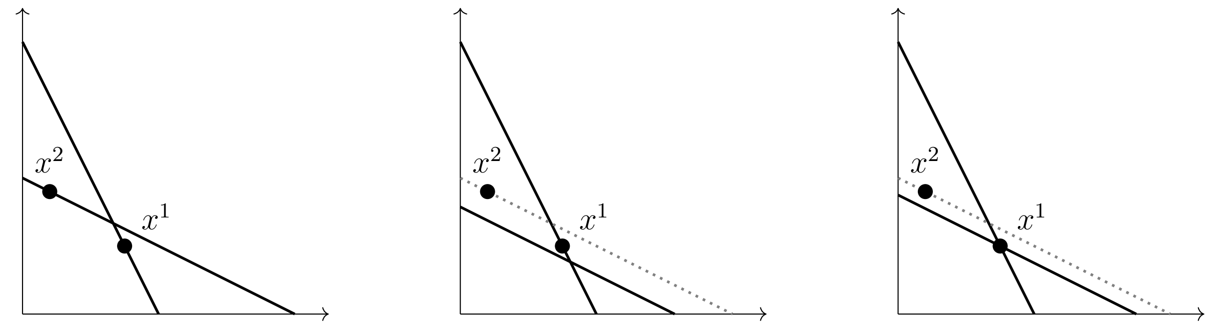
(GARP). In other words, I study the possibility of using an agent's (possibly inconsistent) observed choices to conclude that her preferences are not strictly convex. The main result of the paper shows that if the data fails GARP then it is impossible to test for strict convexity of the underlying preference relation.

Since Afriat (1967) we know that a consumer's choice data can be thought as coming from a preference relation if and only if it satisfies GARP. Matzkin and Richter (1991) shows that we can choose such preference to be strictly convex if and only if it satisfies Houthakker's (1950) Strong Axiom of Revealed Preferences (SARP). However, if for some reason the consumer makes suboptimal choices according to her preferences, i.e., presents bounded rationality, GARP and SARP may be insufficient to recover her preferences.

Halevy et al. (2018) propose a method to recover preferences under bounded rationality. First they show a modified version of Afriat's Theorem that integrates partial efficiency, an idea first proposed by Afriat (1973). Intuitively partial efficiency requires a choice to be preferred not to every feasible alternative but only to those whose cost is a share of the consumer's income. Formally, take a dataset of N observations, where each observation i is a price vector p^i and a choice x^i; partial efficiency $v_i \in [0, 1]$ in choice i requires x^i to be preferred only to bundles whose cost is $v_i \, p^i \, x^i$ instead of $p^i \, x^i$ (if $v_i = 1$ for all the observations, then we go back to the classical, i.e., full efficiency definition of GARP and SARP). Using this idea they propose to recover preferences by, according to a cost function, choosing the partial-efficiency levels that satisfy (a partial-efficiency version of) GARP at a minimum cost.

This paper extends the work in Halevy et al. (2018) to analyze strict convexity of the preferences in this setting. First we show that the equivalence between SARP and strict convexity of the preference relation does not hold under partial efficiency. Specifically, the partial efficiency version of SARP is a sufficient but not necessary condition to rationalize the data with a strictly convex preference relation.

Our main result shows that if partial efficiency is needed to rationalize the data (i.e., if the data fails GARP) then strict convexity cannot be tested. Specifically, if the data fails GARP then for any preference that rationalizes the data (under partial efficiency) we are able to choose another preference that (1) rationalizes the choices (under partial efficiency), (2) is strictly convex, and (3) yields the same cost. An intuitive explanation of this result can be found in Figure 1: in Panel (a) we see a dataset that is not efficient as it fails GARP: x^1 is revealed strictly preferred to x^2 and x^2 is revealed strictly preferred to x^1. We rationalize the data by including partial efficiency in one choice; we do it to x^2 since it requires a smaller shrink of the budget set (the cost of x^1 when x^2 is chosen is a higher share of the income than the cost of x^2 when x^1 is chosen). If we shrink the budget set of x^2 such that x^1 is outside this new budget set, then the data satisfies both GARP and SARP, as x^1 is revealed preferred to x^2 and x^2 is not revealed preferred to x^1. However, whenever x^1 is in the (modified) budget set of x^2 the data will fail both GARP and SARP, even if x^1 is in the boundary of this budget set.



(a) Data fails GARP and SARP

(b) Data satisfies GARP and SARP under partial efficiency

(c) Data fails GARP and SARP under partial efficiency

Figure 1: An intuitive explanation of our main result. As GARP does not hold we need partial efficiency in order to rationalize the data. As long as x^1 is not in the (relaxed) budget set of x^2 (panel (b)) the data satisfies both GARP and SARP. However, if x^1 is within the budget set of x^2 (panel (c)) both GARP and SARP fail.

From Afriat (1967) and Matzkin and Richter (1991) we know that if the data satisfies GARP strict convexity of the preferences can be falsified only if it fails SARP. Our results complete this test by adding that whenever the data fails GARP strict convexity cannot be falsified. Having a complete test we try to falsify the existence of a demand function using experimental data from 322 individuals (50 choices each). For not even one of them are we able to rule out the existence of a demand function, which suggests that this widespread assumption does not carry a substantial cost.

1.1 Related Literature

The idea of revealed preferences traces back to Samuelson (1938). Afriat's (1967) seminal paper shows that observed choices can be thought as generated by a continuous, monotone, and convex preference relation if and only if they satisfy an easy-to-check condition called cyclical consistency, whose most famous version GARP was stated by Varian (1982). Studying further properties of the preferences that can explain the choices, Chiappori and Rochet (1987) shows that the preferences can be chosen to be strictly convex and generating an infinitely differentiable utility function if the data satisfies what they call the Strong version of the Strong Axiom. Focusing solely on strict convexity, Matzkin and Richter (1991) show that SARP is equivalent to strict convexity of preferences. Revealed preferences analysis has been extended in several directions, notably by Forges and Minelli (2009) studying non-linear budget sets, Reny (2015) studying infinite datasets, and Nishimura et al. (2017) studying general choice environments and different criteria for objectively better bundles.

The literature studying datasets that are not rationalizable, i.e., that fail GARP, starts with Afriat (1973), who proposes to use the same level of partial efficiency in all observations to measure distance from economic rationality. After him, several other measures have been proposed, all of them noticing that different observations can use different efficiency levels (Houtman & Maks, 1985; Varian, 1990; Echenique et al., 2011; Dean & Martin, 2016). Polisson et al. (2020) uses the same idea to study distance from expected-utility models. Halevy et al. (2018) take a further step and investigate how to use these measures to recover preferences, focusing specifically in Varian's (1990) Index. The analysis in Halevy et al. (2018) is the starting point of this paper.

The rest of the paper proceeds as follows. Section 2 presents the problem and analyzes whether we can recover strictly convex preferences for a given level of partial efficiency. Section 3 shows how to use the Varian Index to choose the level of partial efficiency, and characterizes our test for strict convexity of the preference relation. Finally, Section 4 concludes.

2 Data Rationalization

2.1 Setup

Consider an agent who consumes bundles of K commodities and has to make N different choices.[1] In each round $i \in [N]$ she faces a price vector $p^i \in \mathbb{R}^K_{++}$, and has to buy a

[1]In the paper we work with the following notation and terminology: $\mathbb{N}$ denotes the set of natural numbers (excluding zero), and $\mathbb{R}$ the set of real numbers. $\mathbb{R}_+$ is the set of positive numbers including zero, and $\mathbb{R}_{++}$ excludes it. For any natural number $M \in \mathbb{N}$ we denote by $[M]$ the set of the first M natural numbers. A vector $\mathbf{y} \in \mathbb{R}^M$ is $\mathbf{y} = (y_1, y_2, \ldots, y_M)$. The vectors $\mathbf{0}$ and $\mathbf{1}$ have all their components equal to zero and one, respectively. For any two vectors $x, y \in \mathbb{R}^M$ we write $x \geq y$ if $x_i \geq y_i$ for all $i \in [M]$, $x > y$ if $x \geq y$ and $x \neq y$, and $x \gg y$ if $x_i > g_i$ for all $i \in [M]$. A function $f : \mathbb{R}^M \to \mathbb{R}$ is order-preserving [order-reversing] if $x > y$ implies $g(x) > [<] \, g(y)$. $||x - y||$ is the Euclidean distance between x and y.

The asymmetric component of a preference relation $\succsim$ is $\succ$. A preference relation is monotone if $x > y$ implies $x \succ y$; is continuous if for all $x \in \mathbb{R}^K_+$ the sets $\{y : x \succsim y\}$ and $\{y : y \succsim x\}$ are closed; and is convex [strictly convex] if $x \succsim y$ implies $\alpha x + (1-\alpha) y \succsim [\, \succ \,]\, y$ for all $\alpha \in (0, 1)$. For a binary relation $\succsim$ we write $(x, y) \notin \succsim$ as $x \not\succsim y$.

bundle whose cost does not exceed her income, which (without loss) we normalize to one. Each price p^i generates the budget set $\{x \in \mathbb{R}^K_+ : p^i x \leq 1\}$, and we denote the choice x^i. Together, prices and choices form the dataset $\mathcal{D} = (x^i, p^i)_{i \in [N]}$, which is the primitive of our problem. We refer to the chosen bundles in the dataset as choices. As standard in the revealed preference literature we assume that the agent spends all her income in each round, i.e., $p^i x^i = 1$ for all $i \in [N]$.

From Afriat (1967) and Varian (1982) we know that we can interpret the choices in $\mathcal{D}$ as coming from the maximization of a locally nonsatiated preference relation if and only if the dataset satisfies GARP, and we can always choose such preference to be monotone, continuous, and convex. Furthermore, from Matzkin and Richter (1991) we know that we can choose such preference to be strictly convex if and only if it satisfies SARP, which is stronger than GARP. Strict convexity of the preference relation $\succsim$ implies that the consumer's demand is a function instead of a correspondence. This is, that for any price vector p there is a unique optimal bundle $x^\star$, i.e., for any affordable $x \neq x^\star$ we have $x^\star \succ x$.[2] The existence of a demand function is a widely used assumption in economics, hence our focus will be on the strict convexity of the recovered preference relation.

If a dataset fails GARP we know that there is no preference relation consistent with the choices. In order to recover preferences in this case, Halevy et al. (2018) use the idea of partial efficiency to relax the requirements of GARP. Specifically, they require each choice x^i to be preferred to bundles whose cost at prices p^i is only a share $v_i \in [0,1]$ of the income. The collection of all such shares is the N-dimension vector $\mathbf{v} = (v_1, \ldots, v_n)$, and the revealed preferences are defined accordingly.

Definition 1. Take a vector $\mathbf{v} \in [0,1]^N$, a choice x^i, and a bundle $x \in \mathbb{R}^K_+$. x^i is
- $\mathbf{v}$-*directly revealed preferred* to x, denoted $x^i \succsim^D_{\mathbf{v}} x$, if $x^i = x$ or $p^i\, x \leq v_i$;
- $\mathbf{v}$-*directly revealed strictly preferred* to x, denoted $x^i \succ^D_{\mathbf{v}} x$, if $p^i\, x < v_i$;
- $\mathbf{v}$-*revealed preferred* to x, denoted $x^i \succsim^R_{\mathbf{v}} x$, if there exists a sequence of choices $(x^{k_\ell})_{\ell=1}^L$, $k_\ell \in [N]$, such that $x^i \succsim^D_{\mathbf{v}} x^{k_1} \succsim^D_{\mathbf{v}} x^{k_2} \succsim^D_{\mathbf{v}} \ldots \succsim^D_{\mathbf{v}} x^{k_L} \succsim^D_{\mathbf{v}} x$; and
- $\mathbf{v}$-*revealed strictly preferred* to x, denoted $x^i \succ^R x$, if there exist choices x^m, $x^{m'}$ such that $x^i \succsim^R_{\mathbf{v}} x^m \succ^D_{\mathbf{v}} x^{m'} \succsim^R_{\mathbf{v}} x$.

The revealed preference relation defined above compares each choice x^i only with bundles affordable at prices p^i and income v_i (instead of 1). As v_i decreases, the set of

Unless stated otherwise all the proofs are in the Appendix.

[2]To see this suppose towards a contradiction that there is an affordable x satisfying $x \succsim x^\star$. By strict convexity for $\alpha \in (0,1)$ we have $\alpha x + (1-\alpha)x^\star \succ x^\star$. But as x is affordable then $p\, x \leq 1$, which implies that $p\,(\alpha x + (1-\alpha)x^\star) \leq 1$. Hence $\alpha x + (1-\alpha)x^\star$ is an affordable bundle that is strictly preferred to $x^\star$ and $x^\star$ is not optimal.

bundles that we compare x^i with shrinks, decreasing the possibility of interpreting x^i as being preferred to another bundle. When $\mathbf{v} = \mathbf{1}$ Definition 1 is equivalent to the classical definition of revealed preferences, and a smaller $\mathbf{v}$ implies a lower level of efficiency in each round. As with the classical definition of GARP we are interested in whether the data we observe can be thought as coming from a (meaningful) complete and transitive preference relation.[3]

Definition 2. A dataset is $\mathbf{v}$-*rationalizable* if there exists a preference relation $\succsim$ such that $x^i \succsim x$ whenever $x^i \succsim^D_{\mathbf{v}} x$. We say that such preference relation $\mathbf{v}$-*rationalizes* the data.

The idea of $\mathbf{v}$-revealed preferences leads to the following definition of data consistency.

Definition 3. Given $\mathbf{v} \in [0,1]^N$ a dataset satisfies the *generalized axiom of revealed preferences given* $\mathbf{v}$ (GARP$_\mathbf{v}$) if for every pair of choices x^i, x^j

$$x^i \succsim^R_{\mathbf{v}} x^j \implies x^j \not\succ^D_{\mathbf{v}} x^i \,.$$

When $\mathbf{v} = \mathbf{1}$ Definitions 2 and 3 are equivalent to the classical definitions of rationalization and GARP, respectively. Hence, we refer to GARP$_\mathbf{1}$ simply as GARP.

From Halevy et al. (2018) we know that Afriat's (1967) theorem can be extended to partial efficiency according to $\mathbf{v}$. This is, they show that a dataset satisfies GARP$_\mathbf{v}$ if and only if it is $\mathbf{v}$-rationalized by a monotone, continuous, and convex preference relation. The next question explores when is it possible for such preference to be strictly convex.

2.2 Strict Convexity of Preferences

From Matzkin and Richter (1991) we know that the existence of a monotone, continuous, and strictly convex preference relation that rationalizes the data is equivalent to Houthakker's (1950) SARP. In the same spirit of Definition 3 we propose an extended definition of SARP that allows for partial efficiency.

Definition 4. Given a vector $\mathbf{v} \in [0,1]^N$ the dataset $\mathcal{D}$ satisfies the *Strong Axiom of Revealed Preferences given* $\mathbf{v}$ (SARP$_\mathbf{v}$) if for every two choices x^i, x^j, whenever $x^i \neq x^j$

$$x^i \succsim^R_{\mathbf{v}} x^j \implies x^j \not\succsim^D_{\mathbf{v}} x^i$$

[3]The trivial preference relation of indifference between all bundles is always consistent with economic rationality.

Again, $\text{SARP}_{\mathbf{1}}$ is equivalent to the original Houthakker's (1950) axiom and hence we refer to it as SARP. The following result shows that although $\text{SARP}_{\mathbf{v}}$ only compares different bundles, it does not present inconsistencies regarding two observations with the same choice.

Fact 1. *If $\mathcal{D}$ satisfies* $\text{SARP}_{\mathbf{v}}$ *and* $x^i = x^j$*, then* $x^i \not\succ^R_{\mathbf{v}} x^j$*.*

A smaller vector $\mathbf{v}$ implies that we interpret each choice as preferred only to cheaper bundles, which reduces the set of revealed preferences. A consequence of this is that the requirements for $\text{SARP}_{\mathbf{v}}$ are relaxed as $\mathbf{v}$ decreases. In the limit case $\mathbf{v} = \mathbf{0}$ the requirements disappear.

Fact 2. *Let* $\mathbf{v}' \leq \mathbf{v}$*. If $\mathcal{D}$ satisfies* $\text{SARP}_{\mathbf{v}}$ *then it satisfies* $\text{SARP}_{\mathbf{v}'}$*.*

Fact 3. *Every dataset $\mathcal{D}$ satisfies* $\text{SARP}_{\mathbf{0}}$*.*

Surprisingly, Matzkin and Richter's (1991) equivalence between SARP and rationalization by a strictly convex preference relation does not hold when we include partial efficiency. Specifically, a strictly convex preference relation could $\mathbf{v}$-rationalize a dataset that fails $\text{SARP}_{\mathbf{v}}$. An example is presented as follows and shown in Figure 2.

Example 1. Suppose $K = 2$ and the dataset $\mathcal{D}$ has two observations, with $p^1 = (1/2, 1/4)$, $x^1 = (9/5, 2/5)$, $p^2 = (1/4, 1/2)$, and $x^2 = (2/5, 9/5)$. Take $\mathbf{v} = (13/20, 13/20)$ and the preference relation $\succsim$ represented by the utility function $U(x) = \sqrt{(1 + x_1)(1 + x_2)}$. As U is strictly concave $\succsim$ is strictly convex. It is easy to check that $\succsim$ $\mathbf{v}$-rationalizes the data. Moreover $x^1 \succsim^D_{\mathbf{v}} x^2$ and $x^2 \succsim^D_{\mathbf{v}} x^1$, which is a violation of $\text{SARP}_{\mathbf{v}}$.

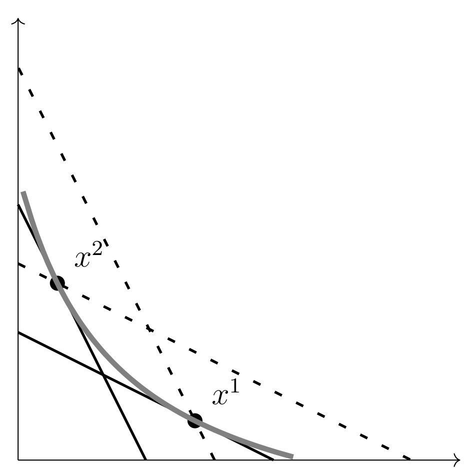


Figure 2: A strictly convex preference that $\mathbf{v}$-rationalizes the data but fails $\text{SARP}_{\mathbf{v}}$.

The intuition for the previous result can be better explained starting from the proof of why **1**-rationalization by strictly convex preferences implies SARP, and then see why the same reasoning cannot be applied when $\mathbf{v} < 1$. When $\mathbf{v} = \mathbf{1}$ the proof proceeds by counterpositive. If the data fails both SARP and GARP then it cannot be rationalized, hence it cannot be rationalized by a strictly convex preference. If it fails SARP but satisfies GARP there are x^i, x^j such that $x^i \neq x^j$, $x^i \succsim^R_{\mathbf{1}} x^j$, $x^j \succsim^D_{\mathbf{1}} x^i$, and $x^j \not\succ^D_{\mathbf{1}} x^i$. This implies that $p^j x^i = v_j = 1$. According to the revealed preference relation, the decision maker is indifferent between x^i and x^j. But as $p^j x^j = p^j x^i = 1$, then for $\alpha \in (0,1)$ the bundle $x^\star = \alpha x^i + (1 - \alpha x^j)$ also satisfies $p^j x^\star = 1$. To rationalize the data by a strictly convex $\succsim$ is impossible since strict convexity implies $x^\star \succ x^j$. Now suppose $\mathbf{v} < 1$ and we have a violation of SARP$_\mathbf{v}$ but not GARP$_\mathbf{v}$: $x^i \succsim^R_\mathbf{v} x^j$ and $p^j x^i = v_j$. If $v_j < 1$, then $x^\star$ does not satisfy $p^j x^\star \leq v_j$ for any $\alpha \in (0,1)$. This is, $x^\star$ is not affordable at prices p^j if the income share of observation j is less than one. Hence we cannot rule out **v**-rationalization by a strictly convex preference relation.

The main result of this section shows that under partial efficiency sufficiency in Matzkin and Richter (1991) still holds. This is, SARP$_\mathbf{v}$ is a sufficient condition for, under partial efficiency, being able to choose a strictly convex preference driving the choices, and hence for the existence of a demand function. Furthermore, as in GARP, SARP, and GARP$_\mathbf{v}$, such preference relation can be chosen to be also continuous and monotone.

Theorem 1. *Take a dataset $\mathcal{D}$ and $\mathbf{v} \in [0,1]^N$. If $\mathcal{D}$ satisfies* SARP$_\mathbf{v}$ *then it is* **v**-*rationalizable by a continuous, monotone, and strictly convex preference relation.*

The proof of this result is included in the Online Appendix. It extends the original proof in Matzkin and Richter (1991): first we show the existence of a modified version of Afriat numbers[4] and then uses those numbers to construct a continuous, order-preserving, and strictly concave utility function representing a preference relation that **v**-rationalizes the data. Given the properties of the utility function this preference is continuous, monotone, and strictly convex.

The next section focuses on how to choose a criterion to pick the level of partial efficiency, i.e., how to pick the vector **v**, and whether we can distinguish between GARP$_\mathbf{v}$ and SARP$_\mathbf{v}$ with such criterion.

3 Partial Efficiency and Preference Recoverability

[4]We show that there are numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ such that $u^i = u^j$ whenever $x^i = x^j$ and $u^i > u^j + \lambda^i(v_i - p^i x^j)$ whenever $x^i \neq x^j$.

3.1 Choosing a partial efficiency level

When the data fails GARP (SARP) we can think of the decision maker as choosing according to a continuous, (strictly) convex, and monotone preference relation only if we allow for partial efficiency. However, for any dataset there is a continuum of vectors $\mathbf{v}$ for which it satisfies $\text{GARP}_{\mathbf{v}}$ ($\text{SARP}_{\mathbf{v}}$), and since there is not a clear order between vectors in $[0,1]^N$, we need a criterion to choose a specific $\mathbf{v}$. Varian (1990) proposes to use a vector $\mathbf{v}$ that is as close a possible to $\mathbf{1}$ in some norm. We denote the distance between $\mathbf{v}$ and $\mathbf{1}$ in such norm by $f(\mathbf{v})$. The only two requirements that we impose to this function are to favor bigger vectors over smaller ones (to be order reversing) and for the costs to be similar when two vectors are close to each other (to be continuous). We also normalize it such that $f(\mathbf{0}) = 1$.

Definition 5. For a dataset $\mathcal{D}$ of N observations let $f : [0,1]^N \to [0,1]$ be a continuous, order-reversing function satisfying $f(\mathbf{1}) = 0$ and $f(\mathbf{0}) = 1$. The *Varian Inefficiency Index* $V(\mathcal{D})$ is defined as

$$V(\mathcal{D}) = \inf_{\{\mathbf{v}\in[0,1]^N : \mathcal{D} \text{ satisfies } \text{GARP}_{\mathbf{v}}\}} f(\mathbf{v})\,. \tag{1}$$

We refer to the Varian Inefficiency Index as simply the Varian Index and to f as the cost function.

The Varian Index is not the only possible criteria to choose the level of partial efficiency. However it appears to be the most suitable for it. As discussed in Halevy et al. (2018) both Afriat's (1973) Critical Cost Efficiency Index and the Index by Houtman and Maks (1985) can be thought as special cases of the Varian Index.[5] Moreover, both these indices will result in a lower efficiency, i.e. a lower $\mathbf{v}$, and hence a higher cost. Another alternative is the Minimum Cost Index proposed by Dean and Martin (2016), but that measure may count twice the cost of shrinking one budget set (if shrinking it opens two violations of GARP), and when it does not it reduces to a special case of the Varian Index. Finally, the Money Pump Index (Echenique et al., 2011) takes the average level of partial efficiency needed to satisfy GARP instead of the minimum level, hence it is not a criterion to choose the vector $\mathbf{v}$.

[5] Afriat's (1973) CCEI imposes for all the components of the vector $\mathbf{v}$ to have the same value. The Houtman and Maks (1985) imposes that each component has to be either zero or one. The CCEI remains the most popular one in the literature, mostly because the Varian Index is computationally more demanding and there are not widespread algorithms to compute it (Smeulders et al. (2014) show that computing the Varian Index is an NP-Hard problem). Halevy et al. (2018) and Polisson et al. (2020) have computed the Varian Index for datasets of moderate size.

3.2 Preference Recoverability

Our main question is how to use the Varian Index to recover a preference relation which we can interpret as driving the choices (under partial efficiency), and whether we can pick such preferences to generate a demand function. The recovery of preferences can be used (among other options) to understand the costs of parametric assumptions (Halevy et al., 2018; Zrill, 2020), to measure welfare, and to obtain information for normative criteria in individual decision-making (Silverman & Kariv, 2013).

We start by analyzing the additional cost of imposing the existence of a demand function. We find that there is no cost at all. That is, if we modify the Varian Index and ask the data to satisfy SARP$_{\mathbf{v}}$ instead of GARP$_{\mathbf{v}}$ it does not change the value of the index.

Proposition 1.

$$V(\mathcal{D}) = \inf_{\{\mathbf{v}\in[0,1]^N : \mathcal{D} \text{ satisfies SARP}_{\mathbf{v}}\}} f(\mathbf{v})\,.$$

The fact that the Varian Index is the same for convex and strictly convex preference relations does not imply that any vector $\mathbf{v}$ recovered with the Varian Index allows us to choose both types of preferences. The natural approach to recover preferences would be to find $\mathbf{v}$ satisfying $f(\mathbf{v}) = V(\mathcal{D})$ (which exists by the intermediate value theorem) and then to either (1) find a specific preference relation that $\mathbf{v}$-rationalizes the data or (2) to bound the set of such preferences.[6] However as the Varian Index is an infimum it might be the case that there is no $\mathbf{v}$ for which $V(\mathcal{D}) = f(\mathbf{v})$ and the data satisfies GARP$_{\mathbf{v}}$, and the same applies to SARP$_{\mathbf{v}}$. The next result shows that whenever partial efficiency is required a vector $\mathbf{v}$ satisfying $f(\mathbf{v}) = V(\mathcal{D})$ cannot be used to recover preferences.

Proposition 2. *If the data fails* GARP [SARP] *then for any* $\mathbf{v}$ *satisfying* $f(\mathbf{v}) = V(\mathcal{D})$ *it also fails* GARP$_{\mathbf{v}}$ [SARP$_{\mathbf{v}}$] .

Figure 1 shows a simple example with an intuitive explanation of Proposition 2. In this case we have $x^1 \succ^D_{\mathbf{1}} x^2$ and $x^2 \succ^D_{\mathbf{1}} x^1$, which is a violation of SARP and GARP. Assume without loss that $f((1, p^2x^1)) < f((p^1x^2, 1))$, i.e., that it is less costly to shrink the budget set of the second observation. As for every $\varepsilon > 0$ small enough we have that $x^2 \not\succsim^R_{(1,p^2x^1-\varepsilon)} x^1$, the data satisfies GARP$_{(1,p^2x^1-\varepsilon)}$. Hence $V(\mathcal{D}) = f((1, p^2x^1))$. Moreover, the data also satisfies SARP$_{(1,p^2x^1-\varepsilon)}$. Finally, if we set $\varepsilon = 0$ we have both $x^1 \succ^D_{(1,p^2x^1)} x^2$ and $x^2 \succsim^D_{(1,p^2x^1)} x^1$, which is a violation of both GARP$_{(1,p^2x^1)}$ and SARP$_{(1,p^2x^1)}$.

[6] Apparently this is the approach taken in Halevy et al. (2018) and Zrill (2020).

Although preferences cannot be recovered by a partial efficiency vector satisfying $f(\mathbf{v}) = V(\mathcal{D})$, they can be recovered with a vector whose cost is arbitrarily close to $V(\mathcal{D})$. Moreover, from Proposition 1 it follows that the same applies when we require strict convexity.

Corollary 1. *For every $\varepsilon > 0$ there is a vector $\mathbf{v}$ such that $f(\mathbf{v}) - V(\mathcal{D}) < \varepsilon$ for which the data satisfies* $\text{SARP}_{\mathbf{v}}$.

Our main result fully characterizes the test to falsify the existence of a demand function: the only case in which this assumption can be falsified is if the data satisfies GARP and fails SARP.

Theorem 2. *Take $\mathbf{v}$ such that the data satisfies* $\text{GARP}_{\mathbf{v}}$. *The only case in which it is impossible to find $\mathbf{v}^\star$ such that $f(\mathbf{v}^\star) = f(\mathbf{v})$ and* $\text{SARP}_{\mathbf{v}^\star}$ *holds is if $\mathbf{v} = 1$ and the data fails* SARP.

Theorem 2 implies that if the data fails GARP, for every vector $\mathbf{v}$ for which the data is $\mathbf{v}$-rationalizable there is another vector $\mathbf{v}^\star$ that yields the same cost as $\mathbf{v}$ and for which we can find a strictly convex preference $\mathbf{v}^\star$-rationalizing the data. This implies that whenever the data fails GARP the partial efficiency criterion does not allow testing for strict convexity. Hence the existence of a demand function can be falsified only in a very specific case: the data has to satisfy GARP but not SARP. Ironically, this implies that the Varian Index is useless to test for strict convexity, as if the data satisfies GARP and fails SARP we know from the classical results by Afriat (1967) and Matzkin and Richter (1991) that the preference relation rationalizing the data cannot be strictly convex.

The final question we address is how usual it is to be able to falsify strict convexity in the data. Theoretically the answer to this question will depend both on the data generating process (DGP) of the price vectors that generate the budget sets, and of the DGPs generating the choice in each budget set. For example, for any dataset in which the budget sets are all different and the choice in each budget set is a continuous random variable we know that to have two different observations i, j satisfying $p^i x^j = 1$ is a zero probability event. This implies that (almost surely) any dataset satisfying GARP will also satisfy SARP, and therefore convexity cannot be tested.

We finish the paper by empirically analyzing whether we can test the existence of a demand function. We analyze 322 subjects, each one making 50 different choices under the experimental design of Choi et al. (2007). Specifically, subjects choose between Arrow securities for three different states of the world. We study choices with three different states of the world ($K = 3$) because it is impossible to identify GARP from SARP in choices with two goods if all the choices are made from different price vectors.[7] In each of the 50 choices the computer randomly selected a budget set, subject to the constraints that all components of the price vector are greater than $1/100$ (all intercepts lie between 0 and 100) and at least one of them is less than $1/50$ (one intercept is greater than 50). Of the total sample 168 subjects knew that all the states had equal probability. The rest are from the experiment in Ahn et al. (2014), in which subjects knew that the first state had probability $1/3$ but did not know the probabilities of the other two states (besides the fact that they added to $2/3$). At the end of the experiment one choice and one state of the world were randomly chosen, and the subject received a payment according to the amount of securities she bought. No subject in our sample satisfies GARP and fails SARP. That is, we cannot rule strict convexity of the preference relation (and hence the existence of a demand function) for any of them.

The specificity of the case in which strict convexity of the preference relation can be tested, along with the fact that we do not observe this case in the data, suggest that under the approach of partial efficiency the widespread assumption of strict convexity of the preference relation does not entail a high cost. We interpret this as a strong signal that working with demand functions should not be a reason for concern in applied economic research, both theoretical and empirical.

4 Final Remarks

One of the most widespread assumptions in economic research, both theoretical and empirical, is the strict convexity of the preference relation. Its main implication is that for any price that the consumer faces there is a unique bundle that is optimal according to her preferences, and hence her demand is a function instead of a correspondence. In this paper we study the possibility of empirically testing this assumption.

From Afriat (1967) and Matzkin and Richter (1991) we know that when a dataset is rationalizable, i.e., when it satisfies GARP, we can test strict convexity of the preferences using SARP: if the data does not pass this test, then the consumer's preference relation cannot be strictly convex. We expand this analysis by recovering preferences through a *partial efficiency* approach, the most popular tool to deal with datasets that fail GARP in the revealed preference literature. Our main result shows that when the data fails GARP strict convexity cannot be falsified; that is, we can always pick the recovered preferences to be strictly convex (therefore to generate a demand function).

[7] Specifically if $K = 2$ and two observations $i, j \in [N]$ are such that $p^i \neq p^j$ and $p^i x^j = 1$, then $x^i = x^j$.

We test the existence of demand functions in an experimental dataset and find that this assumption cannot be falsified in any of the 322 subjects. We interpret this as a strong signal that this widely used assumption does not carry a significant cost.

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Appendix: Proofs

Facts

Proof of Fact 1. By counterpositive suppose $x^i \succ^R_{\mathbf{v}} x^j$. Then there are $m, m' \in [N]$ such that $x^i \succsim^R_{\mathbf{v}} x^m \succ^D_{\mathbf{v}} x^{m'} \succsim^R_{\mathbf{v}} x^j$. As $p^m x^m = 1$ and $v_m \leq 1$ then $x^m \neq x^{m'}$. Furthermore as $x^i = x^j$ we have $x^j \succsim^D_{\mathbf{v}} x^i$, which implies $x^{m'} \succsim^R_{\mathbf{v}} x^m$. As $x^m \succsim^D_{\mathbf{v}} x^{m'}$ $\text{SARP}_{\mathbf{v}}$ does not hold. □

Proof of Fact 2. As $\mathbf{v}' \leq \mathbf{v}$ whenever $x^i \succsim^D_{\mathbf{v}'} x$ we have $x^i \succsim^D_{\mathbf{v}} x$, hence $x^i \succsim^R_{\mathbf{v}'} x \implies x^i \succsim^R_{\mathbf{v}} x$. Suppose $\text{SARP}_{\mathbf{v}}$ holds and $x^i \succsim^R_{\mathbf{v}'} x^j$, which implies $x^i \succsim^R_{\mathbf{v}} x^j$. By $\text{SARP}_{\mathbf{v}}$ we have $x^j \not\succsim^D_{\mathbf{v}} x^i$, i.e. $p^j x^i > v_j$. Then $\mathbf{v} \geq \mathbf{v}'$ implies $p^j x^i > v'_j$, i.e. $x^j \not\succsim^D_{\mathbf{v}'} x^i$. Therefore $\text{SARP}_{\mathbf{v}'}$ holds. □

Proof of Fact 3. Since $p^i x^i = 1$ we have $x^i \neq \mathbf{0}$, and hence $p^j x^i > 0 = v_j$. Therefore for $x^i \neq x^j$ we have $x^i \not\succsim^D_{\mathbf{v}} x^j$ and $\text{SARP}_{\mathbf{0}}$ holds. □

Proposition 1

Before proving Proposition 1 we prove the following auxiliary result.

Lemma 1. *If* $\text{GARP}_{\mathbf{v}}$ *holds then there is a sequence* $(\mathbf{v}^n)_{n\in\mathbb{N}}$ *such that*

1. $\mathbf{v}^n \leq \mathbf{v}^{n+1}$ *for all* n*;*

2. $\mathbf{v}^n \to \mathbf{v}$*; and*

3. $\mathcal{D}$ *satisfies* $\text{SARP}_{\mathbf{v}^n}$ *for all* n*.*

Proof. Suppose $\text{GARP}_{\mathbf{v}}$ holds, and let C be the (potentially empty) set of all pairs of observations $(i,j) \in [N] \times [N]$ such that $x^i \neq x^j$, $x^i \succsim^R_{\mathbf{v}} x^j$, and $x^j \succsim^D_{\mathbf{v}} x^i$. As $\mathcal{D}$ satisfies $\text{GARP}_{\mathbf{v}}$ then for every $(i,j) \in C$ we have $x^j \not\succ^D_{\mathbf{v}} x^i$, which implies $p^j x^i = v_j > 0$ (since $x^i > 0$ and $p^j \gg 0$). Define $\mathbf{v}^n$ by

$$v^n_j = \begin{cases} \frac{n}{n+1} v_j & \text{if } (i,j) \in C \text{ for some } i \in [N] \\ v_j & \text{otherwise.} \end{cases}$$

It is clear that $\mathbf{v}^n \leq \mathbf{v}^{n+1}$ for all n and that $\mathbf{v}^n \to \mathbf{v}$. Moreover, if $(i,j) \in C$ for some $i \in [N]$ then $v^n_j < v_j$.

To see that $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{v}^n}$ suppose $x^i \neq x^j$ and $x^i \succsim^R_{\mathbf{v}^n} x^j$. If $(i,j) \notin C$ then $x^j \not\succsim^D_{\mathbf{v}} x^i$, hence $p^j x^i > v_j \geq v^n_j$, and if $(i,j) \in C$ then $p^j x^i = v_i > v^n_i$. Hence $x^j \not\succsim^D_{\mathbf{v}^n} x^i$ and $\text{SARP}_{\mathbf{v}^n}$ holds. □

Proof of Proposition 1. As $\text{SARP}_{\mathbf{v}}$ is stronger than $\text{GARP}_{\mathbf{v}}$ we have

$$\{\mathbf{v} \in [0,1]^N : \mathcal{D} \text{ satisfies SARP}_{\mathbf{v}}\} \subset \{\mathbf{v} \in [0,1]^N : \mathcal{D} \text{ satisfies GARP}_{\mathbf{v}}\}$$

and therefore

$$\inf_{\{\mathbf{v} \in [0,1]^N : \mathcal{D} \text{ satisfies SARP}_{\mathbf{v}}\}} f(\mathbf{v}) \geq V(\mathcal{D})\,. \tag{2}$$

By definition of $V(\mathcal{D})$ there is a sequence $\mathbf{v}^n \to \mathbf{v}^\star$ such that $\text{GARP}_{\mathbf{v}^n}$ holds for all n and $f(\mathbf{v}^\star) = V(\mathcal{D})$. Furthermore, by Lemma 1 for each n there is a sequence $(\mathbf{b}^{n,i})_{i \in \mathbb{N}}$ such that $\mathbf{b}^{n,i} \to \mathbf{v}^n$ and $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{b}^{n,i}}$ for every $i, n \in \mathbb{N}$. Set $\varepsilon > 0$ and for each n take an element $j(n)$ of $(\mathbf{b}^{n,i})_{i \in \mathbb{N}}$ such that $||\,\mathbf{v}^n - \mathbf{b}^{n,j(n)}|| < \varepsilon/n$. Define the sequence $(\mathbf{c}^n)_{n \in \mathbb{N}}$ by $\mathbf{c}^n = \mathbf{b}^{n,j(n)}$. Then $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{c}^n}$ for all n and $\mathbf{c}^n \to \mathbf{v}^\star$. As f is continuous then $f(\mathbf{c}^n) \to f(\mathbf{v}^\star)$ and hence

$$\inf_{\{\mathbf{v} \in [0,1]^N : \mathcal{D} \text{ satisfies SARP}_{\mathbf{v}}\}} f(\mathbf{v}) \leq f(\mathbf{v}^\star) = V(\mathcal{D})\,. \tag{3}$$

(2) and (3) imply the desired result. □

Proposition 2

We prove Proposition 2 using a criteria of *almost* data consistency, which we take from Polisson et al. (2020) (Appendix A9.1).

Definition 6. For $\mathbf{v} \in [0,1]^N$ the dataset $\mathcal{D}$ *almost satisfies* $\mathrm{GARP}_{\mathbf{v}}$ (i.e., it satisfies $\mathrm{aGARP}_{\mathbf{v}}$) if there is a sequence of vectors $(\mathbf{v}^n)_{n\in\mathbb{N}}$ such that

1. $\mathbf{v}^n \leq \mathbf{v}$;
2. $\mathbf{v}^n \to \mathbf{v}$; and
3. $\mathcal{D}$ satisfies $\mathrm{GARP}_{\mathbf{v}^n}$ for all $n \in \mathbb{N}$.

Lemma 2 presents an alternative characterization of $\mathrm{aGARP}_{\mathbf{v}}$.

Lemma 2. *The data satisfies* $\mathrm{aGARP}_{\mathbf{v}}$ *if and only if there is no sequence of choices* $(x^{m_\ell})_{\ell\in[L]}$ *such that* $x^{m_1} \succ^D_{\mathbf{v}} x^{m_2} \succ^D_{\mathbf{v}} \ldots \succ^D_{\mathbf{v}} x^{m_L}$ *and* $x^{m_L} \succ^D_{\mathbf{v}} x^{m_1}$.

Proof. For necessity suppose such sequence exists. Then

$$p^{m_1}\, x^{m_2} < v_{m_1}\,,\; p^{m_2}\, x^{m_3} < v_{m_2}\,, \ldots, p^{m_{L-1}}\, x^{m_L} < v_{m_{L-1}}\,, \text{ and } p^{m_L}\, x^{m_1} < v_{m_L}\,.$$

Take any sequence $\mathbf{v}^n \to \mathbf{v}$. Then for n large enough

$$p^{m_1}\, x^{m_2} < v^n_{m_1}\,,\; p^{m_2}\, x^{m_3} < v^n_{m_2}\,, \ldots, p^{m_{L-1}}\, x^{m_L} < v^n_{m_{L-1}}\,, \text{ and } p^{m_L}\, x^{m_1} < v^n_{m_L}\,.$$

As $x^{m_1} \succsim^R_{\mathbf{v}^n} x^{m_L}$ and $x^{m_L} \succ^D_{\mathbf{v}^n} x^{m_1}$ the data fails $\mathrm{GARP}_{\mathbf{v}^n}$. As $\mathbf{v}^n$ is arbitrary the data fails $\mathrm{aGARP}_{\mathbf{v}}$.

For sufficiency suppose no sequence exists. Define $\mathbf{v}'$ by

$$v'_i = \begin{cases} 0 & \text{if } \{j \in [N] : p^i\, x^j < v_i\} = \emptyset \\ \max\limits_{\{j\in[N]: p^i\, x^j < v_i\}} p^i\, x^j & \text{otherwise.} \end{cases}$$

Define the sequence $(\mathbf{v}^n)_{n\in\mathbb{N}}$ by $\mathbf{v}^n = \frac{n}{n+1}\,\mathbf{v} + \frac{1}{n+1}\,\mathbf{v}'$, where (as $\mathbf{v}' \leq \mathbf{v}$) $\mathbf{v}^n \leq \mathbf{v}$. Since there are no cycles of directly revealed strict preferences under $\mathbf{v}$ and $\mathbf{v}' \leq \mathbf{v}$ then there are no such cycles under $\mathbf{v}'$ and therefore neither under $\mathbf{v}^n$. As for all n all the $\mathbf{v}^n$-revealed preferences are strict then $\mathrm{GARP}_{\mathbf{v}^n}$ holds. As $\mathbf{v}^n \to \mathbf{v}$ this implies that $\mathrm{aGARP}_{\mathbf{v}}$ holds. □

Lemma 3. *If $\mathcal{D}$ satisfies* $\mathrm{aGARP}_{\mathbf{v}}$*, then $V(\mathcal{D}) \leq f(\mathbf{v})$.*

Proof. By $\mathrm{aGARP}_{\mathbf{v}}$ there is a sequence $\mathbf{v}^n \to \mathbf{v}$ such that $\mathcal{D}$ satisfies $\mathrm{GARP}_{\mathbf{v}^n}$ for all n. If $f(\mathbf{v}) < V(\mathcal{D})$, by continuity of f for n large enough we have $f(\mathbf{v}^n) < f(\mathbf{v}) + \frac{1}{2}(V(\mathcal{D}) - f(\mathbf{v})) < V(\mathcal{D})$, a contradiction with $\mathcal{D}$ satisfying $\mathrm{GARP}_{\mathbf{v}^n}$. □

Proof of Proposition 2. Suppose GARP does not hold and take $\mathbf{v}$ such that $f(\mathbf{v}) = V(\mathcal{D})$. If $\mathcal{D}$ fails $\mathrm{aGARP}_{\mathbf{v}}$ then it also fails $\mathrm{GARP}_{\mathbf{v}}$. If it satisfies $\mathrm{aGARP}_{\mathbf{v}}$ then

- If $\mathbf{v} = \mathbf{1}$ then $\mathrm{GARP}_{\mathbf{v}}$ does not hold by assumption.

- If $\mathbf{v} < \mathbf{1}$ then there is i such that $v_i < 1$; define $A = \{j \in [N] : x^j \succ^R_{\mathbf{v}} x^i\}$. Towards a contradiction suppose GARP$_{\mathbf{v}}$ holds. Then $p^i x^j > v_i$ for all $j \in A$.[8] As A is finite there is ε small enough such that $p^i x^j > v_i + \varepsilon$ for all $j \in A$. Define $\mathbf{v}' \in [0,1]^N$ by

$$v'_n = \begin{cases} v_i + \varepsilon & \text{if } n = i \\ v_n & \text{otherwise.} \end{cases}$$

  Then the directly revealed strict preferences are the same under $\mathbf{v}$ and $\mathbf{v}'$. By Lemma 2 aGARP$_{\mathbf{v}}$ implies that there are no cycles of directly revealed strict preferences under $\mathbf{v}$, and hence neither under $\mathbf{v}'$, i.e. aGARP$_{\mathbf{v}'}$ holds. But $\mathbf{v}' > \mathbf{v}$ implies $f(\mathbf{v}') < f(\mathbf{v}) = V(\mathcal{D})$, which contradicts Lemma 3.

Now suppose SARP does not hold and take $\mathbf{v}$ satisfying $f(\mathbf{v}) = V(\mathcal{D})$.

- If GARP holds then $\mathbf{v} = \mathbf{1}$ and SARP$_{\mathbf{v}}$ does not hold.
- If GARP fails by the previous result GARP$_{\mathbf{v}}$ fails as well. As SARP$_{\mathbf{v}}$ is stronger than GARP$_{\mathbf{v}}$ the data fails SARP$_{\mathbf{v}}$.

□

Corollary 1

Proof of Corollary 1. Take $\varepsilon > 0$. By Proposition 1 there is a sequence $(\mathbf{v}^n)_{n\in\mathbb{N}}$ such that $\mathcal{D}$ satisfies SARP$_{\mathbf{v}^n}$ for all n and $f(\mathbf{v}^n) \to V(\mathcal{D})$. Infinitely many elements of the sequence satisfy $f(\mathbf{v}^n) - V(\mathcal{D}) < \varepsilon$. □

Theorem 2

Proof of Theorem 2. If $\mathbf{v} = 1$ then for any $\mathbf{v}' \neq \mathbf{v}$ we have $f(\mathbf{v}') < 1$, so we need $\mathbf{v}^\star = 1$. From Matzkin and Richter (1991) we know that we can choose the preference relation to be strictly convex if and only if the data satisfies SARP.

If $\mathbf{v} < 1$ from Proposition 2 we have $f(\mathbf{v}) < V(\mathcal{D})$. Let $\varepsilon = f(\mathbf{v}) - V(\mathcal{D})$. By Corollary 1 there is $\mathbf{v}'$ such that $\mathcal{D}$ satisfies SARP$_{\mathbf{v}'}$ and $f(\mathbf{v}') < f(\mathbf{v})$. As f is continuous, order-reversing, and $f(\mathbf{0}) = 1$ there is $\alpha \in [0,1)$ such that $f(\alpha\,\mathbf{v}') = f(\mathbf{v})$. Set $\mathbf{v}^\star = \alpha\,\mathbf{v}'$. As $\mathbf{v}^\star \leq \mathbf{v}'$ SARP$_{\mathbf{v}^\star}$ holds (Fact 2), and by Theorem 1 there is a strictly convex preference relation $\mathbf{v}^\star$-rationalizing the data. □

[8] If $p^i x^j \leq v_i$ for some $j \in A$ we have $x^i \succsim^D_{\mathbf{v}} x^j$. As $x^j \succ^R_{\mathbf{v}} x^i$ there are m, m' such that $x^j \succsim^R_{\mathbf{v}} x^m \succ^D_{\mathbf{v}} x^{m'} \succsim^R_{\mathbf{v}} x^i$. Then $x^{m'} \succsim^R_{\mathbf{v}} x^m$ and $x^m \succ^D_{\mathbf{v}} x^{m'}$, which violates GARP$_{\mathbf{v}}$.

The generality of the Strong Axiom
Online Appendix

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Theorem 1. *Take a dataset $\mathcal{D}$ and $\mathbf{v} \in [0,1]^N$. If $\mathcal{D}$ satisfies* SARP$_\mathbf{v}$ *then it is* $\mathbf{v}$-*rationalizable by a continuous, monotone, and strictly convex preference relation.*

Proof of Theorem 1

Before proving Theorem 1 we show two auxiliary results. The first one shows that under SARP there is a choice that is not revealed preferred to any other one. The second one shows that SARP$_\mathbf{v}$ implies the existence of a modified version of the Afriat numbers.

Lemma 1. *If the dataset satisfies* SARP$_\mathbf{v}$ *there exist $i \in [N]$ such that $p^i x^j > v_i$ for all $x^j \neq x^i$.*

Proof. By counterpositive suppose that for all $i \in [N]$ there exist $x^j \neq x^i$ such that $p^i x^j \leq v^i$. Then we can create an infinite sequence of choices $(x^{j_\ell})_{\ell=1}^{\infty}$ such that for all ℓ we have $x^{j_\ell} \neq x^j_{\ell+1}$ and $x^{j_\ell} \succsim_\mathbf{v}^D x^{j_{\ell+1}}$. As there $N < \infty$ there has to be $m \in [N]$ and $\ell', \ell'' \in \mathbb{N}$ such that $x^m = x^{j'_\ell} = x^{j_{\ell''}}$ and $\ell'' \geq \ell' + 2$. Then $x^m \succsim_\mathbf{v}^D x^{\ell'+1}$ and $x^{\ell'+1} \succsim_\mathbf{v}^R x^{\ell''} = x^m$, which contradicts SARP$_\mathbf{v}$. □

Lemma 2. *If $\mathcal{D}$ satisfies* SARP$_\mathbf{v}$ *then there exist numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ for $i \in [N]$ such that for all $i, j \in [N]$*

$$
\begin{aligned}
&u^i > u^j + \lambda^i (v_i - p^i x^j) && \text{whenever } x^i \neq x^j\text{; and} \\
&u^i = u^j && \text{whenever } x^i = x^j.
\end{aligned}
\tag{1}
$$

Proof. We prove the result by induction on the number of observations N. If $N = 1$, it is clear that $u^1 = \lambda^1 = 1$ satisfy the conditions.

Now suppose the condition holds for all databases with $N-1$ or less observations, and take a database of size N. Take $\varepsilon > 0$ and $m \in [N]$ satisfying $p^m x^i > v_m$ whenever $x^i \neq x^m$ (such m exists by Lemma 1), and write $[N]_m = [N] \setminus \{m\}$. Note that $(p^i, x^i)_{i \in [N]_m}$ is a dataset of $N-1$ observations, and therefore exist $u^i \in \mathbb{R}$ and $\lambda^i > 0$ for all $i \in [N]_m$ such that (1) holds whenever $i, j \in [N]_m$. Define $G = \{i \in [N]_m : x^i \neq x^m\}$ and $B = [N]_m \setminus G$.

- If $B = \emptyset$, set $u^m = \min_{i \in G} u^i + \lambda^i(p^i x^m - v_i) - \varepsilon$. Then for all $i \in G$ we have $u^i > u^m + \lambda^i(v_i - p^i x^m)$.
- If $B \neq \emptyset$, take $\ell \in B$ and set $u^m = u^\ell$. Since for all $i \in G$ condition (1) holds for i, ℓ we have $u^i > u^m + \lambda^i(v_i - p^i x^m)$ if $i \notin B$ and $u^m = u^\ell = u^i$ if $i \in B$.

Set
$$\lambda^m = \max\left\{\max_{i \in G} \frac{u^i - u^m + \varepsilon}{p^m x^i - v_m} \,;\, 1\right\}.$$
Then $\lambda^m \geq 1 > 0$ and $u^m > u^i + \lambda^m(v_m - p^m x^i)$ for all $i \in G$ (since $p^m x^i - v_m > 0$). Therefore (1) holds for all $i \in [N]$. □

Proof of Theorem 1. This proof is based in Lemma 2 of Matzkin and Richter (1991). If $\mathcal{D}$ satisfies SARP$_\mathbf{v}$ by Lemma 2 there are numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ such that (1) holds.

Set $M > 0$, and define

$$g(x) = \left(M + \sum_{k \in [K]} (x_k)^2\right)^{1/2} - M^{1/2}$$
$$\varepsilon_1 = \min_{\{(i,j) \in [N] \times [N] : x^i \neq x^j\}} \frac{u^i - u^j - \lambda^i(v_i - p^i x^j)}{2g(x^i - x^j)}.$$

Then $x^i \neq x^j$ implies $u^i - \varepsilon_1 g(x^i - x^j) > u^j + \lambda^i(v_i - p^i x^j)$. Set

$$\varepsilon = \min\left\{\min_{i \in [N]} \min_{k \in [K]} \frac{\lambda^i p^i_k}{2} \,;\, \varepsilon_1\right\}.$$

Then $\varepsilon > 0$ and

$$u^i - \varepsilon g(x^i - x^j) > u^j + \lambda^i(v_i - p^i x^j) \qquad \text{whenever } x^i \neq x^j \tag{2}$$
$$\lambda^i p^i_k > \varepsilon \qquad \text{for all } i \in [N], k \in [K] \tag{3}$$

For each $i \in [N]$ define the function

$$\phi^i(x) = u^i - \lambda^i(v_i - p^i x) - \varepsilon g(x - x^i),$$

which is continuous, strictly concave, and order-preserving.[1] Define

$$U(x) = \min_{i\in[N]} \phi^i(x)\,.$$

Since $U(\cdot)$ is the minimum of finitely many functions it is also continuous, order-preserving, and strictly convex.

Let m be a minimizer of $\phi^m(x^i)$. If $x^i = x^m$ we have that $u^i = u^m$ and

$$U(x^i) = u^m - \lambda^m(v_m - p^m x^i) - \varepsilon g(x^i - x^m) = u^i - \lambda^m(v_m - p^m x^m) \geq u^i\,.$$

The second equality follows from $g(0) = 0$ and the inequality from $p^m x^m \geq v_m$ and $\lambda^m > 0$. If $x^m \neq x^i$ we have from (2) that

$$U(x^i) = u^m - \lambda^m(v_m - p^m x^i) - \varepsilon g(x^i - x^m) > u^i\,.$$

We conclude that $U(x^i) \geq u^i$.

Define the preference relation $\succsim^\star$ by $x \succsim^\star y \iff U(x) \geq U(y)$. Then $\succsim^\star$ is continuous, strictly monotone, and strictly convex. To show that $\mathcal{D}$ can be $\mathbf{v}$-rationalized by $\succsim^\star$ take $i \in [N]$ and x such that $p^i x \leq v_i$. If $x = x^i$ then $x^i \succsim^\star x$. If $x \neq x^i$ then

$$\begin{aligned}
U(x) &= \min_{j\in[N]} u^j - \lambda^j(v_j - p^j x) - \varepsilon g(x - x^j) \\
&\leq u^i - \lambda^i(v_i - p^i x) - \varepsilon g(x - x^i) && \text{(since } i \in [N]) \\
&\leq u^i - \varepsilon g(x - x^i) && \text{(since } \lambda^i > 0 \text{ and } p^i x \leq v_i) \\
&< u^i && \text{(since } \varepsilon > 0 \text{ and } x \neq x^i) \\
&\leq U(x^i) && \text{(since } U(x^i) \geq u^i).
\end{aligned}$$

Therefore $\succsim^\star$ $\mathbf{v}$-rationalizes the data. □

[1] $\phi^i(\cdot)$ is order-preserving since from (3) for all $k \in [K]$ we have

$$\frac{\partial \phi^i(x)}{\partial x_k} = \lambda^i p^i_k - \varepsilon \left(\frac{(x_k)^2}{M + \sum_{k\in[K]}(x_k)^2} \right) > \lambda^i p^i_k - \varepsilon > 0\,.$$