# The Generality of the Strong Axiom


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**Abstract**

Economic research usually endows consumers with a strictly concave utility function. When choices are rationalizable, this assumption can be tested by the Strong Axiom of Revealed Preferences, SARP, as if they fail such a test, the convexity of the utility is not strict. We extend this test to non-rationalizable choices using partial efficiency, the most popular method to recover preferences. Under partial efficiency, a strictly convex utility cannot be tested. Hence, the existence of a strictly concave utility is falsified if, and only if, choices are rationalizable but fail SARP, which we do not observe in laboratory data. From an empirical standpoint, our results suggest that assuming a strictly concave utility does not carry a cost.



**Keywords:** revealed preferences, single-valued demand, partial efficiency.

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# 1 Introduction

One of the most widespread assumptions in economic research is that agents' behavior can be described using a strictly concave utility function. The main implication of this assumption is that the generated demand is a function instead of a correspondence. This assumption simplifies the analysis in both theoretical and empirical research. Most results in general equilibrium, applied game theory, and mechanism and information design rely upon this assumption to keep the models tractable. Empirically, demand estimation typically proceeds by adding an error term to a parametric demand function.

In this paper, we study the possibility of empirically testing the strict concavity of the utility function. We extend the classical revealed preference analysis to the case when the agent's choices are an imperfect implementation of her preferences, and, therefore, fail the Generalized Axiom of Revealed Preferences (GARP). In other words, we study the possibility of using an agent's (possibly inconsistent) observed choices to falsify the strict concavity of her underlying utility. The main result of this paper shows that if the agent fails GARP, it is impossible to falsify the strict concavity of the utility function.

The Afriat Theorem (Afriat, 1967; Varian, 1982) teaches us that a consumer's choice data can be thought of as perfectly driven by a utility function if and only if it satisfies GARP. However, the utility function recovered with Afriat's method is not strictly concave, generating a demand correspondence instead of a function. Matzkin and Richter (1991) show that a strictly concave utility exists if, and only if, choices satisfy Houthakker's (1950) Strong Axiom of Revealed Preferences (SARP). Moreover, Lee and Wong (2005) show that under SARP, we can always choose such a utility to generate an infinitely differentiable demand. Intuitively, the difference between GARP and SARP is that SARP does not allow for revealed indifference between observed choices.

We study the possibility of falsifying the strict concavity of the utility function under non-rationalizable choices, i.e., choices that fail GARP. Our interpretation of non-rationalizable choices is that the agent has an underlying preference but presents some form of bounded rationality *a la* Simon (1955). We focus on recovering preferences using partial efficiency (Afriat, 1973; Halevy, Persitz, & Zrill, 2018; Varian, 1990), the most popular method to analyze choices that fail GARP non-parametrically.[1] Our main results show that the assumption of a strictly concave utility cannot be falsified when the choices fail to satisfy GARP.

When a consumer's choices are sub-optimal, GARP (and SARP) are insufficient to learn about her preferences. Halevy et al. (2018) propose a method to recover preferences under bounded rationality. Their starting point is the idea of partial efficiency. Intuitively, partial efficiency requires a choice to be preferred only to alternatives whose cost is a given share of the consumer's income instead of to every available alternative. The chosen share of income is the level of partial efficiency. Formally, take a data set of $N$ observations, where each observation $i$ is a price vector $p^i$ and a choice $x^i$; partial efficiency $v_i \in [0, 1]$ in choice $i$ requires $x^i$ to be preferred only to bundles whose cost is $v_i\, p^i\, x^i$ instead of $p^i\, x^i$. The partial efficiency levels of the different observations are collected in the vector $\mathbf{v} = (v_1, \ldots, v_N)$. Halevy et al. (2018) study the possibility of rationalizing observed choices under partial efficiency, which we call $\mathbf{v}$-rationalization in reference to the vector $\mathbf{v}$ of partial efficiency levels. They propose a partial efficiency version of GARP, which we refer to as $\text{GARP}_{\mathbf{v}}$, and show a modified version of the Afriat Theorem. That is, they show that $\mathbf{v}$-rationalization and $\text{GARP}_{\mathbf{v}}$ are equivalent. Using this result, the authors then propose to recover preferences by, according to a cost function, choosing the partial efficiency levels that satisfy $\text{GARP}_{\mathbf{v}}$ at a minimum cost.

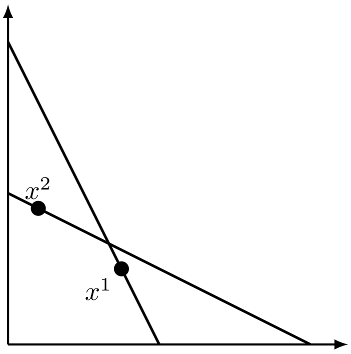
This paper extends the work in Halevy et al. (2018) to analyze the empirical content of the strict concavity of the utility function under partial efficiency. Although under

[1] de Clippel and Rozen (2021) present a more detailed review of different methods.

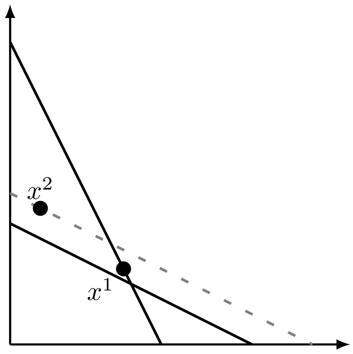
partial efficiency, even a strictly concave utility will generate a demand correspondence, the question of strict concavity sheds light on the empirical content of applied bounded rationality models, as most such models start with a (full efficiency) strictly concave utility, and then add an error term around the demand function it generates.

Our first result shows that $\text{SARP}_{\mathbf{v}}$ (the partial efficiency version of SARP) and $\mathbf{v}$-rationalization by a strictly concave utility are not equivalent. We develop necessary and sufficient conditions for $\mathbf{v}$-rationalization by a strictly concave utility under partial efficiency. These conditions are stronger than $\text{GARP}_{\mathbf{v}}$ but weaker than $\text{SARP}_{\mathbf{v}}$. Put simply, they allow for indifferences between observed choices as long as all choices involved in indifferences are interpreted as sub-optimal (i.e., their partial efficiency level is below one) and each choice has only one choice in the boundary of its (partial efficiency) budget set. As a second result, we show that $\text{SARP}_{\mathbf{v}}$ is necessary and sufficient if $\mathbf{v}$-rationalization is required to be strict, i.e., if every choice is required to be strictly preferred to every other choice in its (partial efficiency) budget set. Additionally, we show that, under $\text{SARP}_{\mathbf{v}}$, the $\mathbf{v}$-rationalizing utility can be chosen to generate (under full efficiency) an infinitely differentiable demand.

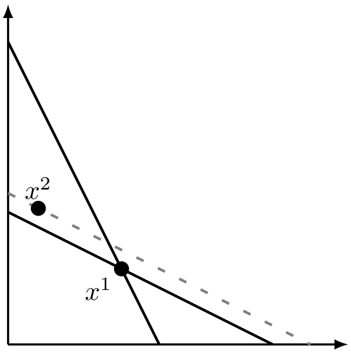
Our main result shows that if the data fails GARP, then, under partial efficiency, the existence of a strictly concave utility cannot be falsified. Specifically, suppose the data fails GARP, then for any utility $u(\cdot)$ and vector $\mathbf{v}$ such that the data can be $\mathbf{v}$-rationalized by $u$, there is another utility $u^\star$ and another vector $\mathbf{v}^\star$ such that (1) the data is $\mathbf{v}^\star$-rationalized by $u^\star$, (2) $u^\star$ is strictly concave, and (3) $\mathbf{v}$ and $\mathbf{v}^\star$ yield the same partial-efficiency loss. Figure 1 presents an intuitive explanation of this result. In (a), we see choice data that fails GARP: $x^1$ is (revealed) strictly preferred to $x^2$, and $x^2$ is (revealed) strictly preferred to $x^1$. To rationalize the data, we need to add partial efficiency to one choice, and we do it to $x^2$ since it requires a smaller shrink of the budget set (the cost of $x^1$ when $x^2$ is chosen is a higher share of the income than the cost of $x^2$ when $x^1$ is chosen). In (b), we shrink the budget set of $x^2$ such that $x^1$




(a) Data fails GARP and SARP




(b) Data satisfies (partial efficiency) GARP and SARP




(c) Data fails (partial efficiency) GARP and SARP

**Figure 1** An intuitive explanation of the main result. In (a) GARP does not hold, so we need partial efficiency to rationalize the data. In (b) $x^1$ is not in the (relaxed) budget set of $x^2$ and both GARP and SARP hold. If $x^1$ is in the upper boundary of $x^2$, as in (c), both GARP and SARP fail.

is outside this new budget set, then the data satisfies both $\text{GARP}_{\mathbf{v}}$ and $\text{SARP}_{\mathbf{v}}$, as $x^1$ is revealed preferred to $x^2$ and $x^2$ is not revealed preferred to $x^1$. However, whenever $x^1$ is in the (partial-efficiency) budget set of $x^2$, even if it is in the upper boundary as in (c), the data will fail both $\text{GARP}_{\mathbf{v}}$ and $\text{SARP}_{\mathbf{v}}$.

From Afriat (1967) and Matzkin and Richter (1991), we know that if the data satisfies GARP, the rationalization by a strictly concave utility can be falsified through SARP. Our results complete this test by adding that recovering a strictly concave utility (via partial efficiency) is always possible whenever the data fails GARP. Having a complete test, we empirically analyze its existence using experimental data from 322 individuals (50 choices each). For none of them we can rule out the strict concavity of the utility function. Such a result suggests that, from an empirical standpoint, this widespread assumption does not carry a cost.

## 1.1 Related Literature

The idea of revealed preferences traces back to Samuelson (1938). Afriat's (1967) seminal paper shows that observed choices can be thought of as generated by a continuous, strictly increasing, and concave utility if, and only if, they satisfy an easy-to-check condition called cyclical consistency. The most famous version of this condition is GARP, proposed by Varian (1982). Matzkin and Richter (1991) show that SARP, a test proposed by Houthakker (1950), is equivalent to a strictly concave utility, therefore generating a demand function. Lee and Wong (2005) strengthen Matzkin and Richter's (1991) result by showing that the same test is sufficient for the utility to generate an infinitely differentiable demand. Revealed preferences analysis has been extended in several directions: Chiappori and Rochet (1987) and Ugarte (2023b) study the differentiability of the utility function, Forges and Minelli (2009) study non-linear budget sets, Reny (2015) studies infinite datasets, and Nishimura, Ok, and Quah (2017) study general choice environments. Dziewulski, Lanier, and Quah (2024) provide a recent review of this literature.

The literature studying non-rationalizable choices, i.e., choices that fail GARP, starts with Afriat (1973). He proposes to use the same level of partial efficiency in all observations to measure the distance from economic rationality. After him, several other measures have been proposed, noticing that different decisions can use different partial efficiency levels (Dean & Martin, 2016; Echenique, Lee, & Shum, 2011; Varian, 1990). Polisson, Quah, and Renou (2020) use the same idea to study distance from expected-utility models. Methods that do not rely on partial efficiency have also been proposed: Houtman and Maks (1985) propose to remove the least number of observations until the remaining data satisfies GARP, and additional methods have been proposed recently (de Clippel & Rozen, 2021; Echenique, Imai, & Saito, 2022; Ugarte, 2023a).[2] Most empirical papers rely on Afriat (1973) or Houtman and Maks (1985) to measure the distance from economic rationality, as these methods are computationally more efficient than others (Demuynck & Rehbeck, 2023).[3] Halevy et al. (2018) take a further step and investigate how to use partial efficiency to recover preferences, focusing specifically on the Varian (1990) Index. The analysis in Halevy et al. (2018) is the starting point of this paper.

The rest of the paper proceeds as follows. Section 2 presents the problem and analyzes conditions to rationalize the choices by a strictly concave utility, given a partial efficiency level. Section 3 shows how to use the Varian Index to choose the level of partial efficiency, characterizes the test for the existence of a strictly concave utility under partial efficiency, and implements this test in laboratory data. Finally, Section 4 concludes.

## 2 Data Rationalization under Partial Efficiency

### 2.1 Setup

Consider an agent who consumes bundles of $K$ commodities and makes $N$ choices.[4] In each choice $i \in [N]$, she faces a price vector $p^i \in \mathbb{R}^K_{++}$ and chooses a bundle $x^i$ from the budget set $\left\{x \in \mathbb{R}^K_+ : p^i x \leq 1\right\}$ (the normalization of income to 1 is without loss of generality). Together, prices and bundles form the data set $\mathcal{D} = (p^i, x^i)_{i\in[N]}$, which

[2] Halevy et al. (2018) discusses how the method proposed by Houtman and Maks (1985) can be interpreted as a special case of partial efficiency.

[3] For example, Fisman, Kariv, and Markovits (2007) apply the Afriat (1973) method, and Caplin, Dean, and Martin (2011) apply the Houtman and Maks (1985) method.

[4] We work with the following notation and terminology: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ the set of real numbers; $\mathbb{R}_+$ is the set of positive numbers including zero, and $\mathbb{R}_{++}$ excludes it. For any $M \in \mathbb{N}$, $[M]$ is the set of the first $M$ natural numbers. A vector $x \in \mathbb{R}^M$ is $x = (x_1, x_2, \ldots, x_M)$, and $||x||$ is its Euclidean norm. The vectors $\mathbf{0}$ and $\mathbf{1}$ have all their components equal to zero and one, respectively. For any two vectors $x, y \in \mathbb{R}^M$ we write $x \geq y$ if $x_i \geq y_i$ for all $i \in [M]$, $x > y$ if $x \geq y$ and $x \neq y$, and $x \gg y$ if $x_i > y_i$ for all $i \in [M]$ ($<$, $\leq$, and $\ll$ are defined similarly). A function $f : \mathbb{R}^M \to \mathbb{R}$ is strictly increasing [strictly decreasing] if $x > y$ implies $g(x) > [<]\, g(y)$. Finally, $\mathbf{1}\{\cdot\}$ is the indicator function.

is the primitive of our problem. We refer to the bundles in $\mathcal{D}$ as choices. As standard in the revealed preference literature, we assume that the agent spends all her income, i.e., $p^i x^i = 1$.

From Afriat (1967) and Varian (1982), we know that we can interpret the choices in $\mathcal{D}$ as coming from the maximization of a locally non-satiated utility if, and only if, $\mathcal{D}$ satisfies GARP. Moreover, we can choose such utility to be strictly increasing, continuous, and concave. However, GARP does not assure the possibility of thinking of the choices as coming from a utility that generates a demand function (instead of a correspondence). Matzkin and Richter (1991) show that $\mathcal{D}$ can be rationalized by a strictly concave utility if, and only if, it satisfies SARP. Strict concavity of the utility function implies that the consumer's demand is a function instead of a correspondence; this is, that for any price vector $p$, there is a unique optimal bundle $x^\star$.[5,6] Taking a further step, Lee and Wong (2005) shows that SARP is also necessary and sufficient for the existence of a utility that generates an infinitely differentiable demand.[7]

If $\mathcal{D}$ fails GARP, no (meaningful) utility function is consistent with the choices.[8] In this case, Halevy et al. (2018) propose to recover a utility function using partial efficiency, a concept proposed by Afriat (1973) and extended by Varian (1990). Partial efficiency requires each choice $x^i$ to be preferred to bundles whose cost at prices $p^i$ is only a share $v_i \in [0,1]$ of the income. The collection of all such shares is the $N$-dimension vector $\mathbf{v} = (v_1, \ldots, v_N)$, and the revealed preferences are defined accordingly.

*Definition* 1. Take $\mathbf{v} \in [0,1]^N$, a choice $x^i$, and a bundle $x \in \mathbb{R}^K_+$. $x^i$ is

- $\mathbf{v}$-*directly revealed preferred* to $x$ (denoted $x^i \succsim^D_{\mathbf{v}} x$) if $x^i = x$ or $p^i x \leq v_i$;

[5]To see this, denote the utility by $U$ and the optimal choice by $x^\star$. By contrapositive take $x \neq x^\star$ satisfying $p\,x \leq 1$ and $U(x) = U(x^\star)$. Let $\alpha \in (0,1)$ and $\hat{x} = \alpha x^\star + (1-\alpha)x$. Then $p\,\hat{x} \leq 1$, and by strict concavity $U(\hat{x}) > U(x^\star)$. Therefore $x^\star$ is not optimal.

[6]Although some utilities are not strictly concave and generate an infinitely differentiable demand (like the Leontieff utility), such cases cannot be identified under linear prices.

[7]If $\mathcal{D}$ fails SARP, the demand is not a function but a correspondence. Thus the classical idea of differentiability does not apply. Although concepts analogous to differentiability have been proposed for correspondences (e.g., Khastan, Rodríguez-López, & Shahidi, 2021), we are not aware of any application of such concepts in economics.

[8]A constant utility always rationalizes $\mathcal{D}$.

- $\mathbf{v}$-*directly revealed strictly preferred* to $x$ ($x^i \succ^D_{\mathbf{v}} x$) if $p^i x < v_i$;
- $\mathbf{v}$-*revealed preferred* to $x$ ($x^i \succsim_{\mathbf{v}} x$) if there exists a sequence of choices $(x^{k_\ell})_{\ell=1}^L$, $k_\ell \in [N]$, such that $x^i \succsim^D_{\mathbf{v}} x^{k_1} \succsim^D_{\mathbf{v}} x^{k_2} \succsim^D_{\mathbf{v}} \dots \succsim^D_{\mathbf{v}} x^{k_L} \succsim^D_{\mathbf{v}} x$;
- $\mathbf{v}$-*revealed strictly preferred* to $x$ ($x^i \succ_{\mathbf{v}} x$) if there exist choices $x^m$, $x^{m'}$ such that $x^i \succsim_{\mathbf{v}} x^m \succ^D_{\mathbf{v}} x^{m'} \succsim_{\mathbf{v}} x$; and
- if $x = x^j$ for some $j \in [N]$, $x^i$ is $\mathbf{v}$-*revealed indifferent* to $x^j$ ($x^i \sim_{\mathbf{v}} x^j$) if $x^i \succsim_{\mathbf{v}} x^j$ and $x^j \succsim_{\mathbf{v}} x^i$.

We write $x^i \not\succsim^D_{\mathbf{v}} x^j$ to denote that $x^i$ is not directly revealed preferred to $x^j$ and use a similar notation for the other revealed preferences.

The revealed preference relations in Definition 1 compare each choice $x^i$ only with bundles affordable at prices $p^i$ and income $v_i \in [0, 1]$, instead of the original income of 1. As $v_i$ decreases, the bundles that we compare $x^i$ with shrink, decreasing the possibility of interpreting $x^i$ as preferred to another bundle. If $\mathbf{v} = \mathbf{1}$, Definition 1 is equivalent to the classical definition of revealed preferences. As with the classical definition of GARP, we are interested in whether the data we observe can be thought of as coming from a (meaningful) utility.

*Definition* 2. $\mathcal{D}$ is $\mathbf{v}$-*rationalizable* by the utility $U : \mathbb{R}^K_+ \to \mathbb{R}$ if $U(x^i) \geq U(x)$ whenever $p^i x \leq v_i$; such utility $\mathbf{v}$-*rationalizes* $\mathcal{D}$. If $U(x^i) > U(x)$ whenever $p^i x \leq v_i$ and $x \neq x^i$, we say that $U$ *strongly* $\mathbf{v}$-*rationalizes* $\mathcal{D}$ (and $\mathcal{D}$ is *strongly* $\mathbf{v}$-*rationalizable* by $U$).

The idea of $\mathbf{v}$-revealed preferences leads to the following definition of data consistency.

*Definition* 3. Take $\mathbf{v} \in [0, 1]^N$. $\mathcal{D}$ satisfies the *Generalized Axiom of Revealed Preferences given* $\mathbf{v}$ (GARP$_{\mathbf{v}}$) if for every pair of choices $x^i$, $x^j$

$$x^i \succsim_{\mathbf{v}} x^j \implies x^j \not\succ^D_{\mathbf{v}} x^i \,.$$

If $\mathbf{v} = \mathbf{1}$, Definition 2 and Definition 3 are equivalent to the classical definitions of rationalization and GARP, respectively. Hence, we refer to $\mathbf{1}$-rationalization as rationalization and to $\text{GARP}_{\mathbf{1}}$ simply as GARP.

From Halevy et al. (2018), we know that Afriat's (1967) theorem can be extended to partial efficiency; this is, $\mathcal{D}$ satisfies $\text{GARP}_{\mathbf{v}}$ if, and only if, it is $\mathbf{v}$-rationalizable by a strictly increasing, continuous, and concave utility. The following section explores when such a utility can be chosen to be strictly concave.

## 2.2 Rationalization by a Strictly Concave Utility

Under full efficiency, Matzkin and Richter (1991) show that $\mathcal{D}$ can be strongly rationalized by a continuous, strictly increasing and strictly concave utility if, and only if, it satisfies SARP. In the same spirit of Definition 3, we propose a partial efficiency version of SARP.

*Definition* 4. Take $\mathbf{v} \in [0,1]^N$. $\mathcal{D}$ satisfies the *Strong Axiom of Revealed Preferences given* $\mathbf{v}$ ($\text{SARP}_{\mathbf{v}}$) if for every two choices $x^i, x^j$, whenever $x^i \neq x^j$

$$x^i \succsim_{\mathbf{v}} x^j \implies x^j \not\succsim_{\mathbf{v}}^D x^i$$

It is easy to see that $\text{SARP}_{\mathbf{v}}$ is equivalent to $\text{GARP}_{\mathbf{v}}$ plus the condition that $x^i \not\sim_{\mathbf{v}} x^j$ whenever $x^i \neq x^j$. Again, $\text{SARP}_{\mathbf{1}}$ is equivalent to Houthakker's (1950) axiom, and hence we refer to it as SARP. The following remark shows that although $\text{SARP}_{\mathbf{v}}$ only compares different bundles, it does not present inconsistencies regarding two observations with the same choice.

*Remark* 1. If $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{v}}$ and $x^i = x^j$, then $x^i \not\succ_{\mathbf{v}} x^j$.

The proofs of the remarks are in Appendix A. A smaller vector $\mathbf{v}$ implies that we interpret each choice as preferred only to cheaper bundles, which reduces the set

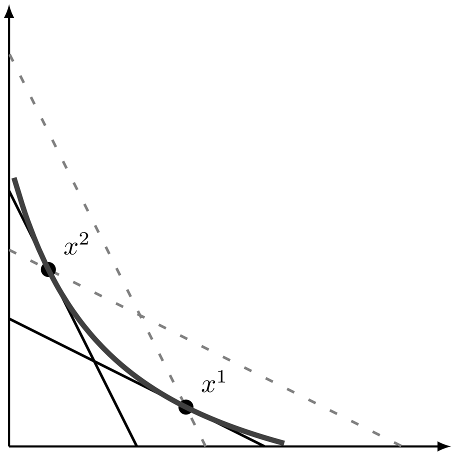



**Figure 2** Representation of Example 1. $\mathcal{D}$ fails SARP$_\mathbf{v}$ but is $\mathbf{v}$-rationalized by a strictly concave and differentiable utility.

of revealed preferences. Consequently, the requirements for SARP$_\mathbf{v}$ are relaxed as $\mathbf{v}$ decreases. In the limit case $\mathbf{v} = \mathbf{0}$, the requirements disappear.

*Remark* 2. Let $\mathbf{v}' \leq \mathbf{v}$. If $\mathcal{D}$ satisfies SARP$_\mathbf{v}$ then it satisfies SARP$_{\mathbf{v}'}$.

*Remark* 3. $\mathcal{D}$ satisfies SARP$_\mathbf{0}$.

Surprisingly, the equivalence between SARP and rationalization by a strictly concave utility does not hold under partial efficiency. Specifically, a strictly concave utility could $\mathbf{v}$-rationalize a data set that fails SARP$_\mathbf{v}$. Furthermore, $\mathbf{v}$-rationalization by a strictly concave utility does not imply strong $\mathbf{v}$-rationalization. A data set that fails SARP$_\mathbf{v}$, and is $\mathbf{v}$-rationalized but not strongly $\mathbf{v}$-rationalized by a strictly concave utility is shown in Example 1 and Figure 2.

*Example* 1. Suppose $K = N = 2$, $p^1 = (1/2, 1/4)$, $x^1 = (9/5, 2/5)$, $p^2 = (1/4, 1/2)$, and $x^2 = (2/5, 9/5)$. Take $\mathbf{v} = (13/20, 13/20)$. As $x^1 \succsim_\mathbf{v} x^2 \succsim_\mathbf{v}^D x^1$, $\mathcal{D}$ fails SARP. The utility function $U(x) = \sqrt{(1 + x_1)(1 + x_2)}$ strictly concave, $\mathbf{v}$-rationalizes $\mathcal{D}$, but does not strongly $\mathbf{v}$-rationalizes it.

The intuition for why SARP$_\mathbf{v}$ is not necessary for $\mathbf{v}$-rationalization by a strictly concave utility can be better understood starting with why failing SARP implies that there is no strictly concave utility rationalizing $\mathcal{D}$ under full efficiency. If $\mathcal{D}$ fails SARP but satisfies GARP, there are $i, j$ such that $x^i \neq x^j$, $x^i \succsim_{\mathbf{1}} x^j$, $x^j \succsim_{\mathbf{1}}^D x^i$, and $x^j \not\succ_{\mathbf{1}}^D x^i$. This implies $p^j x^i = 1$. According to the revealed preference relation, the decision maker is indifferent between $x^i$ and $x^j$. But as $p^j x^j = p^j x^i = 1$, then for $\alpha \in (0,1)$ the bundle $x^\star = \alpha x^i + (1-\alpha)x^j$ satisfies $p^j x^\star = 1$. To rationalize the data by a strictly concave $U$ is impossible as it implies $U(x^\star) > U(x^j)$. Hence $U$ cannot be strictly concave, and the demand has to be a correspondence. Instead, when $\mathbf{v} < 1$, $\text{SARP}_{\mathbf{v}}$ fails, and $\text{GARP}_{\mathbf{v}}$ holds, we have $x^i \succsim_{\mathbf{v}} x^j$ and $p^j x^i = v_j$. If $v_j < 1$, then $x^\star$ does not satisfy $p^j x^\star \leq v_j$ for any $\alpha \in (0,1)$; this is, $x^\star$ is not affordable at prices $p^j$ if the income share of observation $j$ is less than one. Therefore we cannot rule out $\mathbf{v}$-rationalization by a strictly concave utility.

The previous example suggests that necessary and sufficient conditions for $\mathbf{v}$-rationalization by a strictly concave utility are "between" $\text{GARP}_{\mathbf{v}}$ and $\text{SARP}_{\mathbf{v}}$, i.e., are stronger than $\text{GARP}_{\mathbf{v}}$ but weaker than $\text{SARP}_{\mathbf{v}}$. The following result presents such conditions.

**Theorem 1.** *$\mathcal{D}$ is $\mathbf{v}$-rationalizable by a continuous, strictly increasing, and strictly concave utility if, and only if, it satisfies* $\text{GARP}_{\mathbf{v}}$ *and the following conditions hold*

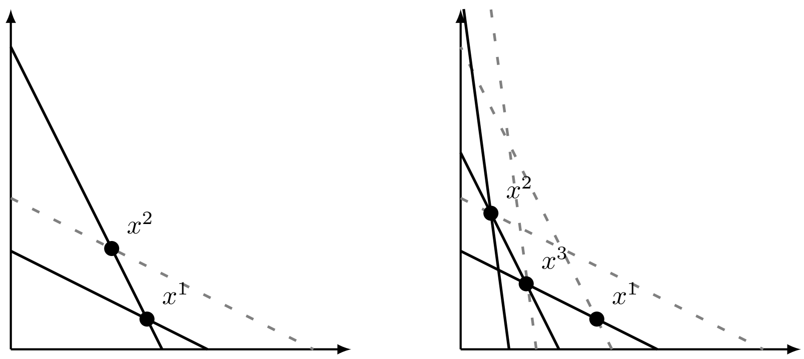
*C1: if $i, j \in [N]$ are such that $x^i \neq x^j$ and $x^i \sim_{\mathbf{v}} x^j$, then $v_i, v_j < 1$; and*

*C2: if $i, j, m \in [N]$ are such that $x^i \sim_{\mathbf{v}} x^j \sim_{\mathbf{v}} x^m$, and $p^i x^j = p^i x^m = v_i$, then $x^m = x^j$.*

The proof of this result is in Appendix B. As in the proofs of Matzkin and Richter (1991), it first shows a modified version of the Afriat inequalities (in which the inequalities are strict), and then constructs the utility function by first taking linear functions (one for each observation, as in the original proof of Afriat (1967)), then introducing a small strict concavification to each of this functions, and finally defining the utility as the pointwise minimum of such functions. However, our proof presents two important distinctions with respect to Matzkin and Richter (1991). The first difference is that the numbers are constructed taking into account the revealed indifferences:

we first generate inequalities between choices that do not reveal indifferences to each other (Lemma 2), and then expand them to generate inequalities also between some observations that reveal indifference to each other. Second, the bundle around which the strict concavification is generated is different; in Matzkin and Richter (1991) each linear function is concavified around the bundle it corresponds to (the bundle in its observation); in our proof we need an additional step to choose where to concavify around.

Figure 3 shows data sets that illustrate the necessity of *C1* and *C2*. In a), the data vacuously satisfies *C2* and fails *C1* as $v_2 = 1$. Since $U(x^1) = U(x^2)$, the **v**-rationalization condition $U(x^2) \geq U(x)$ whenever $p^2x \leq v_2$ implies that all convex combinations $x$ of $x^1$ and $x^2$ mucst satisfy $U(x) \leq U(x^2)$, hence $U$ cannot be strictly concave. In b) we have $v_i < 1$ for all the observations, hence *C1* holds, but $x^1 \sim_{\mathbf{v}} x^2 \sim_{\mathbf{v}} x^3$, $p^2x^1 = p^2x^3 = v_2$, and $x^1 \neq x^3$. Any utility $U$ **v**-rationalizing this data set satisfies $U(x^1) = U(x^2) = U(x^3)$, and the **v**-rationalization condition $U(x^2) \geq U(x)$ whenever $p^2x \leq v_2$ implies that $U$ cannot be strictly concave, as strict concavity would imply $U(z) > U(x^2)$ whenever $z$ is a convex combination of $x^1$ and $x^3$.




(a) Data fails *C1* and satisfies *C2*　　(b) Data satisfies *C1* and fails *C2*

**Figure 3** Data sets that fail *C1* and *C2*, respectively. In both cases, the data cannot be rationalized by a strictly concave utility

Since $\text{SARP}_{\mathbf{v}}$ implies that $x^i \not\sim_{\mathbf{v}} x^j$ when $x^i \neq x^j$, the conditions in Theorem 1 are weaker than $\text{SARP}_{\mathbf{v}}$. Hence, if the data satisfies $\text{SARP}_{\mathbf{v}}$ it can be $\mathbf{v}$-rationalized by a strictly concave utility. The following result shows that $\text{SARP}_{\mathbf{v}}$ adds the property that such rationalization is strict, i.e., that $U(x^i) > U(x)$ whenever $p^i x \leq v_i$ and $x \neq x^i$.

**Theorem 2.** *$\mathcal{D}$ is strongly $\mathbf{v}$-rationalizable by a continuous, strictly increasing, and strictly concave utility if, and only if, it satisfies* $\text{SARP}_{\mathbf{v}}$.

We omit the proof of Theorem 2 as it is similar to the one of Theorem 1. The proof of sufficiency involves only a slight modification to the proof of Theorem 1 using the fact that there are no $\mathbf{v}$-revealed indifferences between different observations. The proof of necessity follows from the fact that that if the data is $\mathbf{v}$-rationalizable by a strictly concave utility but fails $\text{SARP}_{\mathbf{v}}$, then we can find $i, j$ such that $x^i \neq x^j$, $x^i \sim_{\mathbf{v}} x^j$, and $p^i x^j = v_i$. Since the $\mathbf{v}$-revealed preferences imply that $x^i$ and $x^j$ have the same utility level, the $\mathbf{v}$-rationalization is not strict.

### 2.3 Differentiability of the Demand Function

In economic terms, the main distinction between a concave and a strictly concave utility is that, under full efficiency, the demand under the latter is a function while under the former is a correspondence. Lee and Wong (2005) show that, whenever the data satisfies SARP, the utility can be chosen such that it generates a demand that is infinitely differentiable. The following result shows that this property can also be obtained in the case of $\text{SARP}_{\mathbf{v}}$.

**Proposition 1.** *The utility function in Theorem 2 can be chosen to generate (under full efficiency) an infinitely differentiable demand.*

The proof of the previous result is in Appendix C. It starts from the utility function in Theorem 2 and then constructs an auxiliary data set which is $\mathbf{1}$-rationalized by the same utility. Then, following the proof in Lee and Wong (2005), it generates a second utility that shares the properties of $U$ but generates an infinitely differentiable demand. Finally, from the equivalence on how both utilities compare choices in the auxiliary data set, we can conclude that the second utility strongly $\mathbf{v}$-rationalizes $\mathcal{D}$. In Appendix F, we show that if we impose partial efficiency in all observations ($\mathbf{v} \ll \mathbf{1}$), we can further strengthen the result by generating an infinitely differentiable utility.

The following section focuses on choosing a criterion to pick the level of partial efficiency, i.e., how to select the vector $\mathbf{v}$, and whether we can distinguish between a concave and a strictly concave utility under such criterion.

# 3 Testing Strict Concavity

## 3.1 Choosing a partial efficiency level

When $\mathcal{D}$ fails GARP, we can think of the decision maker as choosing according to a meaningful utility only if we allow for partial efficiency. However, for any data set, there is a continuum of vectors $\mathbf{v}$ for which it satisfies $\text{GARP}_{\mathbf{v}}$, and since there is not a clear order between vectors in $[0,1]^N$, we need a criterion to choose a specific $\mathbf{v}$. Varian (1990) proposes to use a vector $\mathbf{v}$ that is as close as possible to $\mathbf{1}$ in some norm, using the quadratic norm as an example. Halevy et al. (2018) formalizes this notion using an aggregator function $f(\mathbf{v})$. The only requirements that we impose on $f(\mathbf{v})$ are to favor bigger vectors over smaller ones (to be strictly decreasing) and for its value to be similar when two vectors are close (to be continuous). We also normalize it such that $f(\mathbf{1}) = 0$ and $f(\mathbf{0}) = 1$.

*Definition* 5. Let $f : [0,1]^N \to [0,1]$ be a continuous and strictly decreasing function satisfying $f(\mathbf{1}) = 0$ and $f(\mathbf{0}) = 1$. The *Varian Inefficiency Index* $V(\mathcal{D})$ is

$$V(\mathcal{D}) = \inf_{\{\mathbf{v}\in[0,1]^N : \mathcal{D} \text{ satisfies } \text{GARP}_{\mathbf{v}}\}} f(\mathbf{v}) \,. \tag{1}$$

We refer to the Varian Inefficiency Index as the Varian Index and to $f$ as the loss function.

The Varian Index is not the only possible criterion for choosing the level of partial efficiency; however, as discussed in Halevy et al. (2018), it is the most suitable for it. Both Afriat's (1973) Critical Cost Efficiency Index (CCEI) and the Houtman and Maks (1985) Index (HM Index) can be thought of as variations of the Varian Index that relax some properties of the loss function.[9] Importantly, both these indices will result in lower efficiency, i.e., a lower $\mathbf{v}$.

An alternative measure of distance from GARP, whose motivation is closely related to the Varian Index, is the Minimum Cost Index (MCI, Dean & Martin, 2016). The MCI is defined as the cost (in terms of partial efficiency) of removing revealed preferences that generate violations of GARP (i.e., that generate revealed preference cycles that include a strict preference) as a share of the total expenditure.[10] Its main conceptual difference with the Varian Index is that reducing the partial efficiency level of one observation might remove several revealed preferences that involve different GARP violations; in this case, the Varian Index interprets it as only one cost, while the MCI interprets it as several cost (one per each violation). To be more specific, suppose we have a data set of three observations, with $p^1x^2 = p^1x^3 = .95$, $p^2x^1 = p^3x^1 = .5$, and $p^2x^3, p^3x^2 > 1$. Under full efficiency, this dataset has two violations of GARP: $x^1 \succ^D_{\mathbf{1}} x^2 \succ^D_{\mathbf{1}} x^1$, and $x^1 \succ^D_{\mathbf{1}} x^3 \succ^D_{\mathbf{1}} x^1$. The Varian Index (under $f(\mathbf{v}) = N^{-1}||\mathbf{1} - \mathbf{v}\,||$ equals $.05/3$, where .05 is the partial efficiency level needed in the first observation to remove both violations of GARP, and 3 is the number of observations. The MCI equals $(2 \cdot .05)/3$, as the partial efficiency level of the first observation is counted twice (one per each violation of GARP). Although the MCI could also be used as a criterion

[9] Afriat's (1973) CCEI imposes for all the components of the vector $\mathbf{v}$ to have the same value (hence $f(\mathbf{v}) = 1 - \min_{i\in[N]} v_i$, which is not increasing), and Houtman and Maks (1985) impose that each component has to be either zero or one ($f(\mathbf{v}) = \sum_{i\in[N]} \mathbf{1}\{v_i < 1\}$, which is neither continuous or strictly increasing). The CCEI remains the most popular in the literature, mainly because the Varian Index is computationally more demanding: Smeulders, Spieksma, Cherchye, and De Rock (2014) show that it is NP-Hard. Recently, Demuynck and Rehbeck (2023) developed mixed-integer linear programming methods to compute the Varian Index and the HM Index and showed that these indices (the Varian Index with a linear loss function) can be quickly computed for datasets regularly collected in experiments.

[10] Let $R$ be the directly revealed preference relation restricted to observed choices, with a typical element being $(x^i, x^j)$. The MCI is defined as $MCI = N^{-1} \min_{B\subset R} \sum_{(x^i,x^j)\in B} 1 - p^i x^j)$ such that $R \setminus B$ is acyclic. The MCI could be used to choose a vector $\mathbf{v}$ by taking $v_i = \min_{(x^i,x^j)\in B^\star} p^i x^j$, where $B^\star$ is the minimizer of the previous problem.

to choose a partial efficiency vector $\mathbf{v}$, we analyze the Varian Index over the MCI because of its popularity in the literature. Furthermore, we suspect the same analysis we do here for the Varian Index might be done to the MCI, reaching similar conclusions.

Finally, another popular measure of distance from GARP is the Money Pump Index (MPI, Echenique et al., 2011). Although the MPI also measures the cost of eliminating GARP violations in terms on partial efficiency, it focuses on the average cost (instead of the minimum cost) of rationalizing a data set and, as such, does not provide a clear criterion to choose a partial efficiency level as it does not yield a cost minimizer vector $\mathbf{v}$.[11]

### 3.2 Preference Recoverability

The main questions of this paper are how to use the Varian Index to recover a utility that we can interpret as driving the choices (under partial efficiency) and whether such utility can be chosen to be strictly convex. This utility can be used, for example, to understand the costs of parametric assumptions (Halevy et al., 2018; Zrill, 2024), measure welfare, and obtain information for normative criteria in individual decision-making (Kariv & Silverman, 2013). For this exercise, we focus our analysis on $\text{SARP}_{\mathbf{v}}$, which, although stronger than $\mathbf{v}$-rationalization by a strictly concave utility, is enough to characterize our result.

We start by analyzing the additional loss of imposing SARP under partial efficiency. We find that there is no loss at all: if we modify the Varian Index and ask $\mathcal{D}$ to satisfy $\text{SARP}_{\mathbf{v}}$ instead of $\text{GARP}_{\mathbf{v}}$, it does not change the value of the index.

**Proposition 2.**

$$V(\mathcal{D}) = \inf_{\{\mathbf{v}\in[0,1]^N : \mathcal{D} \text{ satisfies } \text{SARP}_{\mathbf{v}}\}} f(\mathbf{v})\,.$$

[11] For each cycle $(x^{m_1}, x^{m_2}, \ldots, x^{m_L})$ in the directly revealed preference relation, the MPI of the cycle is $MPI_{(x^{m_1}, x^{m_2}, \ldots, x^{m_L})} = L^{-1} \sum_{\ell\in[L]} 1 - p^{m_\ell} x^{m_{\ell+1}}$, where $x^{m_{L+1}} = x^1$. The authors propose considering the mean and median of MPIs across cycles as measures of distance from GARP.

The proof of this result is in Appendix D. Given that the conditions in Theorem 1 are stronger than $\mathrm{GARP}_{\mathbf{v}}$ and weaker than $\mathrm{SARP}_{\mathbf{v}}$, the following is a direct consequence of the previous result.

**Corollary 1.**

$$V(\mathcal{D}) = \inf_{\{\mathbf{v}\in[0,1]^N : \mathcal{D} \text{ is } \mathbf{v}\text{-rationalizable by a strictly concave utility}\}} f(\mathbf{v})\,.$$

Given the definition of the Varian Index and Proposition 2, the natural approach to recover preferences would be first to find $\mathbf{v}$ satisfying $f(\mathbf{v}) = V(\mathcal{D})$ (which exists by the intermediate value theorem) and then to analyze the utilities that $\mathbf{v}$-rationalize $\mathcal{D}$. However, as the Varian Index is an infimum, it might be the case that there is no $\mathbf{v}$ for which $V(\mathcal{D}) = f(\mathbf{v})$ and $\mathcal{D}$ satisfies $\mathrm{GARP}_{\mathbf{v}}$. The following result, whose proof is in Appendix E, shows that the latter is the case.

**Proposition 3.** *If $\mathcal{D}$ fails* GARP*, then for any $\mathbf{v}$ satisfying $f(\mathbf{v}) = V(\mathcal{D})$ it also fails* $\mathrm{GARP}_{\mathbf{v}}$ *(and hence* $\mathrm{SARP}_{\mathbf{v}}$*).*

Figure 1 shows a simple example that illustrates Proposition 3. In this case we have $x^1 \succ^D_{\mathbf{1}} x^2$ and $x^2 \succ^D_{\mathbf{1}} x^1$, which is a violation of SARP and GARP. Assume without loss that $f((1, p^2x^1)) < f((p^1x^2, 1))$, i.e., that it is less costly to shrink the budget set of the second observation. As for every $\varepsilon > 0$ small enough we have that $x^2 \not\succsim_{(1,p^2x^1-\varepsilon)} x^1$, $\mathcal{D}$ satisfies $\mathrm{GARP}_{(1,p^2x^1-\varepsilon)}$. Hence $V(\mathcal{D}) = f((1, p^2x^1))$. Finally, for $\varepsilon = 0$ we have $x^1 \succ^D_{(1,p^2x^1)} x^2$ and $x^2 \succsim^D_{(1,p^2x^1)} x^1$, hence $\mathrm{GARP}_{(1,p^2x^1)}$ fails.

Even though a partial efficiency vector $\mathbf{v}^\star$ satisfying $f(\mathbf{v}^\star) = V(\mathcal{D})$ cannot recover preferences, they can be recovered using a vector $\mathbf{v}$ that, although smaller than $\mathbf{v}^\star$, is arbitrarily close to it. Our main result, Theorem 3, (which is a direct consequence of Proposition 2 and Proposition 3) shows that the same can be done when the utility is required to be strictly concave.

**Theorem 3.** *For every* $\mathbf{v} < \mathbf{1}$ *such that* $\mathcal{D}$ *is* $\mathbf{v}$*-rationalizable, there is* $\mathbf{v}^\star$ *such that* $f(\mathbf{v}^\star) = f(\mathbf{v})$ *and* $\mathcal{D}$ *is strongly* $\mathbf{v}^\star$*-rationalizable by a continuous, strictly increasing, and strictly concave utility.*

*Proof.* As $\mathbf{v} < 1$ and GARP$_\mathbf{v}$ holds, Proposition 3 implies $f(\mathbf{v}) < V(\mathcal{D})$. By Proposition 2 there is $(\mathbf{v}^n)_{n\in\mathbb{N}}$ such that SARP$_{\mathbf{v}^n}$ holds for all $n$, and $f(\mathbf{v}^n) \to V(\mathcal{D})$. Thus $f(\mathbf{v}^{n_0}) \geq f(\mathbf{v})$ for $n_0$ large enough. As $f$ is continuous and strictly decreasing, there is $\mathbf{v}^\star \leq \mathbf{v}^{n_0}$ such that $f(\mathbf{v}^\star) = f(\mathbf{v})$. Remark 2 implies that $\mathcal{D}$ satisfies SARP$_{\mathbf{v}^\star}$. □

Theorem 3 implies that, if $\mathcal{D}$ fails GARP, then for every vector $\mathbf{v}$ for which it is $\mathbf{v}$-rationalizable there is another vector $\mathbf{v}^\star$ that yields the same cost as $\mathbf{v}$, and for which we can find a strictly concave utility function that $\mathbf{v}^\star$-rationalizes $\mathcal{D}$. Furthermore, Proposition 1 implies that such utility can be chosen such that it generates (under full efficiency) an infinitely differentiable utility. Theorem 3 result fully characterizes the test to falsify the strict concavity of the utility function; specifically, it implies that whenever $\mathcal{D}$ fails GARP, it is impossible to test for this property. Thus, a strictly concave utility can be falsified only in a particular case: $\mathcal{D}$ has to satisfy GARP and fail SARP.

### 3.3 Empirical Implementation

The final question we address is how usual it is to be able to falsify a strictly concave utility. Theoretically, the answer to this question will depend on the data-generating process (DGP) of the price vectors that generate the budget sets and the DGPs generating the choice in each budget set. For example, for any data set in which the budget sets are all different, and the choice in each budget set is a continuous random variable, we know that to have two different observations $i, j$ such that $p^i x^j = 1$ is a zero probability event. Hence (almost surely), any data set satisfying GARP will also satisfy SARP, so convexity cannot be tested.

We empirically analyze the possibility of falsifying the existence of a differentiable demand function using experimental data from 322 subjects, coming from the experiments in Ahn, Choi, Gale, and Kariv (2014) and Dembo, Kariv, Polisson, and Quah (2025). Each subject makes 50 different choices under the design of Choi, Fisman, Gale, and Kariv (2007): they face a budget set to choose Arrow securities for three states of the world. We study choices with three states ($K = 3$) because it is impossible to identify GARP from SARP if there are only two states and all prices differ.[12,13] In each choice, the computer randomly selects a budget set satisfying that all components of the price vector are greater than $1/100$ (all intercepts lie between 0 and 100). At least one of them is less than $1/50$ (one intercept is greater than 50). Of the total sample, the 168 subjects from Dembo et al. (2025) knew that all the states had equal probability. The 154 subjects from Ahn et al. (2014) knew that one state had probability $1/3$ but did not know the probabilities of the other two (besides the fact that they added to $2/3$). At the end of the experiment, the computer randomly chose one choice and one state of the world, and the subject received payment according to the securities she bought.

The main finding of our analysis is that no subject satisfies GARP and fails SARP. Hence, we cannot rule out the existence of a strictly concave utility, neither one generating (under full efficiency) an infinitely differentiable demand. The specificity of the case in which these properties can be tested and that we do not observe it in the data suggest that moving from a concave to a strictly concave utility function does not carry a cost. We interpret this as a strong signal that assuming strict concavity of the utility function should not be a concern in applied economic research.

[12] If $K = 2$ and two observations $i, j \in [N]$ are such that $p^i \neq p^j$ and $p^i x^j = 1$, then $x^i = x^j$.

[13] Dembo et al. (2025) show the superiority of experiments with three instead of two states regarding its power in testing rationalization.

# 4 Final Remarks

One of the most widespread assumptions in economic research, both theoretical and empirical, is to endow consumers with a strictly concave utility function. The main advantage of this assumption is that when consumers choose perfectly according to this function, the generated demand is a function instead of a correspondence. Even research that assumes bounded rationality usually starts from a strictly concave utility and then adds noise to the choices it generates. In this paper, we study the strict concavity of the utility function, focusing on cases when the observed choices are not perfectly aligned with the agent's underlying preferences.

From Afriat (1967) and Matzkin and Richter (1991), we know that for rationalizable choices, i.e., choices that satisfy GARP, SARP is the test for the existence of a strictly concave utility rationalizing the choices. If the data satisfies SARP, then there is a strictly concave utility that rationalizes the data; if it fails SARP, then there is not. We expand this analysis by recovering preferences through partial efficiency, the most popular tool to analyze choices that fail GARP.

We first analyze rationalization by a strictly concave utility under partial efficiency. We characterize this rationalization, showing that under partial efficiency, the conditions are weaker than SARP, as they allow for (some) indifferences between suboptimal choices. Then, we show that SARP is necessary if we require rationalization to be strong; this is, for the choice to be strictly preferred to all the alternatives in its (partial efficiency) budget set. Moreover, we show that under SARP, the utility can be chosen to generate (under full efficiency) an infinitely differentiable demand.

Our main result shows that if choices fail GARP, it is impossible to differentiate between a concave and a strictly concave utility. Using partial efficiency, we can always choose a strictly concave utility that rationalizes the choices at the lowest possible cost. We then test the existence of a strictly concave utility in experimental data and find that this property cannot be falsified in any of the 322 subjects analyzed. Our results suggest that the widely used assumption of a strictly concave utility function demand does not carry an empirical cost, thereby validating its use in economic research.

***Statements and Declarations***

The author has no competing interests (financial or non-financial) related to the research developed in this paper.

# Appendix A Proofs of Remarks

*Proof of Remark 1.* By contrapositive suppose $x^i \succ_{\mathbf{v}} x^j$. Then there are $m, m'$ such that $x^i \succsim_{\mathbf{v}} x^m \succ^D_{\mathbf{v}} x^{m'} \succsim_{\mathbf{v}} x^j$. As $p^m x^m = 1$ and $v_m \leq 1$, then $x^m \neq x^{m'}$. Furthermore, $x^i = x^j$ implies $x^j \succsim^D_{\mathbf{v}} x^i$, thus $x^{m'} \succsim_{\mathbf{v}} x^m$, a violation of SARP$_{\mathbf{v}}$. □

*Proof of Remark 2.* As $\mathbf{v}' \leq \mathbf{v}$, $x^i \succsim^D_{\mathbf{v}'} x$ implies $x^i \succsim^D_{\mathbf{v}} x$ . Hence $x^i \succsim_{\mathbf{v}'} x$ implies $x^i \succsim_{\mathbf{v}} x$. Suppose SARP$_{\mathbf{v}}$ holds and $x^i \succsim_{\mathbf{v}'} x^j$. Then $x^i \succsim_{\mathbf{v}} x^j$, which by SARP$_{\mathbf{v}}$ implies $x^j \not\succsim^D_{\mathbf{v}} x^i$. Hence $x^j \not\succsim^D_{\mathbf{v}'} x^i$ and SARP$_{\mathbf{v}'}$ holds. □

*Proof of Remark 3.* $p^i x^i = 1$ implies $x^i > \mathbf{0}$; hence, as $p^j \gg \mathbf{0}$, we have $p^j x^i > 0 = v_j$. Therefore $x^i \neq x^j$ implies $x^i \not\succsim^D_{\mathbf{v}} x^j$ and SARP$_{\mathbf{0}}$ holds vacuously. □

# Appendix B Proof of Theorem 1

**Lemma 1.** *There is $i \in [N]$ such that $p^j x^m > v_j$ whenever $x^j \sim_{\mathbf{v}} x^i$ and $x^m \not\sim_{\mathbf{v}} x^i$.*

*Proof.* Towards a contradiction suppose for every $i \in [N]$ there are $x^j \sim_{\mathbf{v}} x^i$ and $x^m \not\sim_{\mathbf{v}} x^i$ such that $p^j x^m \leq v_j$. Then $x^i \succsim_{\mathbf{v}} x^j \succsim^D_{\mathbf{v}} x^m$, hence $x^i \succsim_{\mathbf{v}} x^m$. Thus, $x^m \not\sim_{\mathbf{v}} x^i$ implies $x^m \not\succsim_{\mathbf{v}} x^i$ and $x^i \neq x^m$. Hence, we can construct an infinite sequence $(n_\ell)_{\ell \in \mathbb{N}}$ such that, for every $\ell$, $x^{n_\ell} \neq x^{n_{\ell+1}}$ $x^{n_\ell} \succsim_{\mathbf{v}} x^{n_{\ell+1}}$ and $x^{n_{\ell+1}} \not\succsim_{\mathbf{v}} x^{n_\ell}$. As $\mathcal{D}$ is finite, there is an observation that repeats in the sequence, i.e., there are $r, s \in \mathbb{N}$ such that $s \geq r + 2$ and $x^{n_r} = x^{n_s}$. But then $x^{n_{r+1}} \succsim_{\mathbf{v}} x^{n_s} = x^{n_r}$, a contradiction. □

**Lemma 2.** *If* $\text{GARP}_{\mathbf{v}}$ *holds there are numbers* $u^i \in \mathbb{R}$ *and* $\lambda^i > 0$ *such that*

$$\begin{aligned} &u^i > u^j + \lambda^i(v_i - p^i x^j) && \textit{whenever } x^i \not\sim_{\mathbf{v}} x^j\textit{; and} \\ &u^i = u^j \textit{ and } \lambda^i = \lambda^j && \textit{whenever } x^i \sim_{\mathbf{v}} x^j. \end{aligned} \tag{B1}$$

*Proof.* We proceed by induction on $N$. If $N = 1$, then $u^1 = \lambda^1 = 1$.

Suppose GARP holds for all databases comprised of $N-1$ or less observations, and take $\mathcal{D}$ comprised of $N$ observations. By Lemma 1, and without loss of generality, suppose $N$ is such that $p^i \cdot x^j > 1$ whenever $x^i \sim_{\mathbf{v}} x^N$ and $x^j \not\sim_{\mathbf{v}} x^N$. If $x^i \sim_{\mathbf{v}} x^N$ for all $i$, set $u^i = \lambda^i = 1$ for every $i \in [N]$. Then the conditions hold.

If there is $j$ such that $x^j \not\sim_{\mathbf{v}} x^N$, then the data set $(p^j, x^j)_{\{j : x^j \not\sim_{\mathbf{v}} x^N\}}$ is comprised of $N-1$ or less observations, hence numbers $u^i, \lambda^i$ satisfying the conditions exist for this data set. Take $\varepsilon > 0$, and for every $i$ such that $x^i \sim_{\mathbf{v}} x^N$ set

$$u^i = \min_{\{m : x^m \sim_{\mathbf{v}} x^N\}} \min_{\{j : x^j \not\sim_{\mathbf{v}} x^N\}} u^j - \lambda^j(v_j - p^j \cdot x^m) - \varepsilon\,.$$

As $\sim_{\mathbf{v}}$ is an equivalence relation, $u^i = u^j$ whenever $x^i \sim_{\mathbf{v}} x^j$. Moreover, whenever $x^j \not\sim_{\mathbf{v}} x^N$ and $x^i \sim_{\mathbf{v}} x^N$ we have $u^i \leq u^j - \lambda^j(v_j - p^j \cdot x^i) - \varepsilon < u^j - \lambda^j(v_j - p^j \cdot x^i)$. Whenever $x^i \sim_{\mathbf{v}} x^N$ set

$$\lambda^i = \max\left\{ \max_{\{m : x^m \sim_{\mathbf{v}} x^N\}} \max_{\{j : x^j \not\sim_{\mathbf{v}} x^N\}} \frac{u^j - u^m}{p^m \cdot x^j - v_m} + \varepsilon;\, 1 \right\}.$$

Hence $\lambda^i = \lambda^N > 0$ whenever $x^i \sim_{\mathbf{v}} x^N$. Finally, if $x^i \sim_{\mathbf{v}} x^N$ and $x^j \not\sim_{\mathbf{v}} x^N$ it follows form the definition of $\lambda^i$ and $p^i \cdot x^j > v_i$ that $u^i > u^j + \lambda^i(v_i - p^i \cdot x^j)$. □

*Proof of sufficiency in Theorem 1.* Set $M > 0$, and define $g(x) = \left(M + ||x||^2\right)^{1/2} - M^{1/2}$. As $\mathcal{D}$ satisfies $\text{GARP}_{\mathbf{v}}$ there are numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ such that (B1) holds (Lemma 2).

For every $i$, let $I_i = \{j \in [N] : x^i \neq x^j,\, x^i \sim_{\mathbf{v}} x^j, \text{ and } p^i x^j = v_i\}$. Then $I_i = \emptyset$ if, and only if, there is no $x^j \neq x^i$ such that $x^i \sim_{\mathbf{v}} x^j$.[14] Let

$$B_i = \begin{cases} I_i & \text{if } I_i \neq \emptyset \\ \{j \in [N] : x^i = x^j\} & \text{if } I_i = \emptyset \end{cases}$$

As $i \in B_i$ whenever $I_i = \emptyset$, then $B_i \neq \emptyset$ for all $i$ and $i \in B_i$ if, and only if, $I_i = \emptyset$. We show that $u^i > u^j + \lambda^i(v_i - p^i x^j)$ whenever $j \notin B_i$:

- If $I_i = \emptyset$, then from (B1) we have $u^i > u^j + \lambda^i(v_i - p^i x^j)$ whenever $x^j \notin B_i$.
- If $I_i \neq \emptyset$, GARP$_{\mathbf{v}}$ implies $p^i x^j > v_i$ whenever $x^i \neq x^j$, $x^i \sim_{\mathbf{v}} x^j$ and $j \notin B_i$; furthermore, condition *C1* implies $p^i x^j > v_i$ whenever $x^i = x^j$. Since $u^i = u^j$ whenever $x^i \sim_{\mathbf{v}} x^j$ and $\lambda^i > 0$, we have $u^i > u^j + \lambda^i(v_i - p^i x^j)$ whenever $x^i \sim_{\mathbf{v}} x^j$ and $j \notin B_i$. Along with (B1), this implies $u^i > u^j + \lambda^i(v_i - p^i x^j)$ whenever $x^j \notin B_i$.

Define the function $\gamma : [N] \to [N]$ by $\gamma(i) = j$ for some $j \in B_i$, and note that, by *C2*, $x^{\gamma(i)} = x^j$ whenever $j \in B_i$. As $\lambda^i p^i_k > 0$ there is $\varepsilon > 0$ such that

$$u^i - \varepsilon g(x^{\gamma(i)} - x^j) > u^j + \lambda^i(v_i - p^i x^j) \qquad \text{whenever } j \notin B_i; \text{ and} \tag{B2}$$

$$\lambda^i p^i_k > \varepsilon \qquad \text{for all } i \in [N], k \in [K]\,. \tag{B3}$$

Let $\phi^i(x) = u^i - \lambda^i(v_i - p^i x) - \varepsilon g(x - x^{\gamma(i)})$ for every $i \in [N]$, and $U(x) = \min_{i \in [N]} \phi^i(x)$. Since each $\phi^i$ is continuous, strictly concave, and strictly increasing, $U(x)$ inherits these properties.[15]

[14] If there is $x^j \neq x^i$ satisfying $x^i \sim_{\mathrm{v}} x^j$ then $x^i \succsim_{\mathrm{v}} x^j$ and there is $m \in [N]$ such that $x^i \neq x^m$ and $x^i \succsim^D_{\mathrm{v}} x^m \succsim_{\mathrm{v}} x^i$; as GARP$_{\mathrm{v}}$ holds we have $p^i x^m = v_i$, hence $m \in I_i$. If there is not $x^j \neq x^i$ satisfying $x^i \sim_{\mathrm{v}} x^j$, then clearly $I_i = \emptyset$.

[15] $\phi^i(\cdot)$ is strictly increasing since, from (B3), for all $k \in [K]$

$$\frac{\partial \phi^i(x)}{\partial x_k} = \lambda^i p^i_k - \varepsilon \frac{x_k - x^{\gamma(i)}_k}{(M + ||x - x^{\gamma(i)}||)^{1/2}} > \lambda^i p^i_k - \varepsilon > 0\,.$$

We show that $U(x^i) \geq u^i$ for all $i$. If $j \notin B_i$, (B2) implies $\phi^j(x^i) > u^i$. If $j \in B_i$, then $u^i = u^j$ (by (B1)), $p^j \cdot x^i \geq v_j$ (by GARP$_{\mathbf{v}}$), and $x^{\gamma(j)} = x^i$; hence $\phi^j(x^i) = u^i - \lambda^j(v_j - p^j x^i) - \varepsilon g(x^\gamma(j) - x^i) = u^i - \lambda^j(v_j - p^j x^i) \geq u^i$.

Finally, take $x$ satisfying $p^i x \leq v_i$. Then

$$U(x) \leq u^i - \lambda^i(v_i - p^i x) - \varepsilon g(x - x^{\gamma(i)}) \leq u_i \leq U(x^i)\,.$$

The first inequality follows form the definition of $U$; the second from $\lambda^i > 0$, $\varepsilon > 0$, $p^i x \leq v_i$; and $g(\cdot) \geq 0$; and the last from $U(x^i) \geq u^i$. We conclude that $U$ $\mathbf{v}$-rationalizes $\mathcal{D}$. □

*Proof of necessity in Theorem 1.* Suppose that $\mathcal{D}$ is rationalizable by a continuous, strictly increasing, and strictly concave utility.

- That GARP$_{\mathbf{v}}$ holds follows from Theorem 1 in Halevy et al. (2018).
- Towards a contradiction suppose *C1* fails, i.e., there are $i, j \in [N]$ such that $x^i \neq x^j$, $x^i \sim_{\mathbf{v}} x^j$, and, without loss of generality, $v_i = 1$. As $x^i \succsim_{\mathbf{v}} x^j$, there is a sequence of observations $(m_\ell)_{\ell \in [L]}$ such that $x^i \succsim^D_{\mathbf{v}} x^{m_1} \succsim^D_{\mathbf{v}} \ldots \succsim^D_{\mathbf{v}} x^{m_L} \succsim^D_{\mathbf{v}} x^j$, where, without loss of generality, $x^{m_1} \neq x^i$. Furthermore, $x^{m_1} \succsim_{\mathbf{v}} x^j$ and $x^j \succsim^{\mathbf{v}} x^i$ imply $x^{m_1} \succsim_{\mathbf{v}} x^j$. As $\mathcal{D}$ is $\mathbf{v}$-rationalizable we have $U(x^i) = U(x^{m_1})$. Moreover, $x^i \succsim^D_{\mathbf{v}} x^{m_1}$ implies $p^i x^{m_1} \leq 1$, thus $p^i({x^i}/{2} + {x^{m_1}}/{2}) \leq 1$. However, strict concavity of $U$ implies $U({x^i}/{2} + {x^{m_1}}/{2}) > U(x^i)$, which contradicts $\mathbf{v}$-rationalization by $U$.
- Towards a contradiction suppose *C2* fails, i.e., there are $i, j, m \in [N]$ satisfying $x^i \sim_{\mathbf{v}} x^j \sim_{\mathbf{v}} x^m$, $p^i x^j = p^i x^m = v_i$, and $x^j \neq x^m$. Since $U$ $\mathbf{v}$-rationalizes $\mathcal{D}$, $\mathbf{v}$-revealed preferences imply $U(x^i) = U(x^j) = U(x^m)$. By strict concavity of $U$, $U({x^j}/{2} + {x^m}/{2}) > U(x^i)$, which, As $p^i({x^j}/{2} + {x^m}/{2}) = v_i$, contradicts $\mathbf{v}$-rationalization by $U$.

□

# Appendix C Proof of Proposition 1

**Lemma 3.** *Let $U$ be a continuous, strictly concave, and strictly increasing utility generating a demand $m : \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K_+$. For every $x \in \mathbb{R}^K_+ \setminus \{\mathbf{0}\}$ there is $p \gg \mathbf{0}$ such that $m(p, 1) = x$.*

*Proof.* Take $x \in \mathbb{R}^K_+ \setminus \{\mathbf{0}\}$. Since $U$ is strictly concave it has a supergradient $b$ at $x$; this is, $U(x) > U(y) + b(x - y)$ whenever $y \neq x$. We first show, by contradiction, that $b \gg \mathbf{0}$. Suppose $b_k \leq 0$ for some $k \in [K]$, denote by $e^k$ the vector with $k^{\text{th}}$ component equal to one and all the others equal to zero, and set $y = x + e^k > x$. As $U$ is strictly increasing, $U(y) + b(x - y) = U(y) - b\, e^k = U(y) - b_k \geq U(y) > U(x)$, a contradiction.

Define $p = (b\, x)^{-1} b \gg \mathbf{0}$. Then $p\, x = 1$. Moreover, $y \neq x$ and $p\, y \leq 1$ imply $U(x) > U(y)$ (since $b\, x \leq b\, y$). Therefore $x = m(p, 1)$. □

*Proof of Proposition 1.* Suppose $\mathcal{D}$ satisfies SARP$_{\mathbf{v}}$ and let $U$ be a continuous, strictly concave, and strictly increasing utility $\mathbf{v}$-rationalizing $\mathcal{D}$ (Theorem 2). Let $m(p, e)$ be the demand function generated by $U$. By strong $\mathbf{v}$-rationalization, for every $i \in [N]$ we have $U(x^i) \geq U(m(p^i, v_i))$, with strict inequality if $v_i < 1$. Construct the data set $\widetilde{\mathcal{D}} = (\widetilde{p}^j, \widetilde{x}^j)_{j \in [J]}$ as follows: for every $i$ in $[N]$

- If $v_i = 1$, add an observation $(\widetilde{p}^j, \widetilde{x}^j) = (p^i, x^j)$.
- If $v_i < 1$, add two observations $(\widetilde{p}^j, \widetilde{x}^j)$ and $(\widetilde{p}^{j'}, \widetilde{x}^{j'})$, where:
  - $\widetilde{p}^j = p^i$, $\widetilde{x}^j = m(p^i, v_i)$, and
  - $\widetilde{x}^{j'} = x^i$, $\widetilde{p}^{j'} = p$ for some $p$ such that $m(p, 1) = x^i$, which exists by Lemma 3.

By construction $\widetilde{\mathcal{D}}$ is strongly $\mathbf{1}$-rationalized by $U$, hence it satisfies SARP. By Lee and Wong (2005) there is an strictly increasing, strictly concave $\widetilde{U}$ that strongly $\mathbf{1}$-rationalizes $\widetilde{\mathcal{D}}$ and generates an infinitely differentiable demand. Furthermore, from their proof we can choose $\widetilde{U}$ agreeing with $U$ on how to compare choices in $\widetilde{\mathcal{D}}$, i.e.,

$$\widetilde{U}(x^i) \geq \widetilde{U}(x^j) \iff U(x^i) \geq U(x^j) \qquad \text{whenever } i, j \in [J]$$

Finally, take any $i \in [N]$.

- If $v_i = 1$ then there is $j \in [J]$ such that $(\widetilde{p}^j, \widetilde{x}^j) = (p^i, x^i)$. By strong rationalization of $\widetilde{\mathcal{D}}$ we have that $\widetilde{U}(x^i) > \widetilde{U}(x)$ whenever $p^i x \leq v_i$ and $x \neq x^i$.
- If $v_i < 1$ then there are $j, j' \in [J]$ such that $\widetilde{p}^j = p^i$, $\widetilde{x}^j = m(p^i, v_i)$, $\widetilde{x}^{j'} = x^i$, and $m(\widetilde{p}^{j'}, 1) = x^i$. As $v_i < 1$, strong rationalization of $\mathcal{D}$ by $U$ implies $U(x^i) > U(m(p^i, v_i)) = U(\widetilde{x}^j)$. Moreover, as $\widetilde{U}$ and $U$ agree on how to rank choices in $\widetilde{\mathcal{D}}$, we have $\widetilde{U}(x^i) > \widetilde{U}(\widetilde{x}^j)$. Strong rationalization of $\widetilde{\mathcal{D}}$ by $\widetilde{U}$ implies that $\widetilde{U}(x^i) > \widetilde{U}(\widetilde{x}^j) \geq \widetilde{U}(x)$ whenever $p^i x \leq v_i$.

Therefore $\widetilde{U}$ strongly $\mathbf{v}$-rationalizes $\mathcal{D}$.

□

# Appendix D Proof of Proposition 2

**Lemma 4.** *If* $\text{GARP}_{\mathbf{v}}$ *holds then there is a sequence* $(\mathbf{v}^n)_{n \in \mathbb{N}}$ *such that*

1. $\mathbf{v}^n \leq \mathbf{v}^{n+1}$ *for all* $n$*;*
2. $\mathbf{v}^n \to \mathbf{v}$*; and*
3. $\mathcal{D}$ *satisfies* $\text{SARP}_{\mathbf{v}^n}$ *for all* $n$*.*

*Proof.* Suppose $\text{GARP}_{\mathbf{v}}$ holds, and let $C = \{(i,j) \in [N] \times [N] : x^i \neq x^j, x^i \succsim_{\mathbf{v}} x^j, \text{ and } x^j \succsim^D_{\mathbf{v}} x^i\}$. As $\mathcal{D}$ satisfies $\text{GARP}_{\mathbf{v}}$, then $(i,j) \in C$ implies $x^j \not\succ^D_{\mathbf{v}} x^i$, thus $v_j = p^j x^i > 0$. Define $\mathbf{v}^n$ by

$$v_j^n = \begin{cases} \frac{n}{n+1} v_j & \text{if } (i,j) \in C \text{ for some } i \in [N] \\ v_j & \text{otherwise.} \end{cases}$$

Then $\mathbf{v}^n \leq \mathbf{v}^{n+1}$ for all $n$, and $\mathbf{v}^n \to \mathbf{v}$. Moreover, if $(i,j) \in C$ then $v_j^n < v_j$.

Finally, suppose $x^i \neq x^j$ and $x^i \succsim_{\mathbf{v}^n} x^j$. If $(i,j) \notin C$ then $x^j \not\succsim^D_{\mathbf{v}} x^i$, and $p^j x^i > v_j \geq v_j^n$. If $(i,j) \in C$ then $p^j x^i = v_i > v_i^n$. Hence $x^j \not\succsim^D_{\mathbf{v}^n} x^i$, therefore $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{v}^n}$. □

*Proof of Proposition 2.* As $\text{SARP}_{\mathbf{v}}$ is stronger than $\text{GARP}_{\mathbf{v}}$,

$$\inf_{\{\mathbf{v}\in[0,1]^N:\mathcal{D}\text{ satisfies SARP}_{\mathbf{v}}\}} f(\mathbf{v}) \geq V(\mathcal{D})\,. \tag{D4}$$

By definition of $V(\mathcal{D})$ there is a sequence $\mathbf{v}^n \to \mathbf{v}^\star$ such that $\text{GARP}_{\mathbf{v}^n}$ holds for all $n$ and $f(\mathbf{v}^\star) = V(\mathcal{D})$. By Lemma 4, for each $n$ there is a sequence $(\mathbf{b}^{n,i})_{i\in\mathbb{N}}$ such that $\mathbf{b}^{n,i} \to \mathbf{v}^n$ and $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{b}^{n,i}}$ for every $i, n \in \mathbb{N}$. Set $\varepsilon > 0$ and for each $n$ take $j(n)$ such that $||\,\mathbf{v}^n - \mathbf{b}^{n,j(n)}|| < \varepsilon/n$. Define the sequence $(\mathbf{c}^n)_{n\in\mathbb{N}}$ by $\mathbf{c}^n = \mathbf{b}^{n,j(n)}$. Then $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{c}^n}$ for all $n$ and $\mathbf{c}^n \to \mathbf{v}^\star$. Continuity of $f$ implies $f(\mathbf{c}^n) \to f(\mathbf{v}^\star)$, thus

$$\inf_{\{\mathbf{v}\in[0,1]^N:\mathcal{D}\text{ satisfies SARP}_{\mathbf{v}}\}} f(\mathbf{v}) \leq f(\mathbf{v}^\star) = V(\mathcal{D})\,. \tag{D5}$$

(D4) and (D5) imply the desired result. □

## Appendix E Proof of Proposition 3

This proof uses a criteria of *almost* data consistency developed by Polisson et al. (2020, Appendix A9.1).

*Definition* 6. For $\mathbf{v} \in [0,1]^N$, $\mathcal{D}$ *almost satisfies* $\text{GARP}_{\mathbf{v}}$ (i.e., it satisfies $\text{aGARP}_{\mathbf{v}}$) if there is a sequence $(\mathbf{v}^n)_{n\in\mathbb{N}}$ such that

1. $\mathbf{v}^n \leq \mathbf{v}$;
2. $\mathbf{v}^n \to \mathbf{v}$; and
3. $\mathcal{D}$ satisfies $\text{GARP}_{\mathbf{v}^n}$ for all $n \in \mathbb{N}$.

**Lemma 5.** *$\mathcal{D}$ satisfies $\text{aGARP}_{\mathbf{v}}$ if, and only if, when restricted to the choices in $\mathcal{D}$, $\succ^D_{\mathbf{v}}$ is acyclic.*

*Proof.* See Polisson et al. (2020, Appendix A9.1). □

**Lemma 6.** *If $\mathcal{D}$ satisfies $\text{aGARP}_{\mathbf{v}}$, then $V(\mathcal{D}) \leq f(\mathbf{v})$.*

*Proof.* It follows from the definitions of $V(\mathcal{D})$ and aGARP$_{\mathbf{v}}$. □

*Proof of Proposition 3.* Suppose GARP fails and take $\mathbf{v}$ such that $f(\mathbf{v}) = V(\mathcal{D})$. If $\mathcal{D}$ fails aGARP$_{\mathbf{v}}$ then it also fails GARP$_{\mathbf{v}}$. If it satisfies aGARP$_{\mathbf{v}}$ then

- If $\mathbf{v} = \mathbf{1}$ then GARP$_{\mathbf{v}}$ fails by assumption.
- If $\mathbf{v} < \mathbf{1}$ then there is $i$ such that $v_i < 1$; define $A = \{j \in [N] : x^j \succ_{\mathbf{v}} x^i\}$. Towards a contradiction suppose GARP$_{\mathbf{v}}$ holds. Then $p^i x^j > v_i$ for all $j \in A$.[16] As $A$ is finite there is $\varepsilon > 0$ such that $p^i x^j > v_i + \varepsilon$ for all $j \in A$. Define $\mathbf{v}' \in [0,1]^N$ by

$$v'_n = \begin{cases} v_i + \varepsilon & \text{if } n = i \\ v_n & \text{otherwise.} \end{cases}$$

  When restricted to choices in $\mathcal{D}$, $\succ^D_{\mathbf{v}} = \succ^D_{\mathbf{v}'}$. By Lemma 5, aGARP$_{\mathbf{v}}$ implies that $\succ^D_{\mathbf{v}}$ restricted to choices is acyclic, hence $\succ^D_{\mathbf{v}'}$ also is and aGARP$_{\mathbf{v}'}$ holds. But $\mathbf{v}' > \mathbf{v}$ implies $f(\mathbf{v}') < f(\mathbf{v}) = V(\mathcal{D})$, which contradicts Lemma 6.

As $\mathcal{D}$ fails GARP$_{\mathbf{v}}$, it also fails SARP$_{\mathbf{v}}$. □

# Appendix F Differentiable Utility under partial efficiency

Here we address the problem of strongly $\mathbf{v}$-rationalizing a data set using a utility function that is continuous, strictly increasing, strictly concave, and infinitely differentiable. We show that under SARP$_{\mathbf{v}}$, it is possible to recover an infinitely differentiable utility if we impose partial efficiency in all observations, i.e., if $\mathbf{v} \ll \mathbf{1}$. As shown by Chiappori and Rochet (1987), a sufficient condition for differentiability of a strictly concave utility function is for the demand data to be "invertible"; this is, different choices have to come from different prices ($p^i \neq p^j$ implies $x^i \neq x^j$). A non-invertible

[16] If not, then $x^i \succsim^D_{\mathbf{v}} x^j$. As $x^j \succ_{\mathbf{v}} x^i$ there are $m, m'$ such that $x^j \succsim_{\mathbf{v}} x^m \succ^D_{\mathbf{v}} x^{m'} \succsim_{\mathbf{v}} x^i$. Then $x^{m'} \succsim_{\mathbf{v}} x^m$ and $x^m \succ^D_{\mathbf{v}} x^{m'}$, and GARP$_{\mathbf{v}}$ fails.

demand implies that the same bundle is optimal from two different price vectors, which (as income is normalized to one) contradicts the first order condition of equality between the marginal rate of substitution and the price ratio.[17] However, if the partial efficiency level of a choice is less than one, this equality is not necessary anymore. Hence we can find a differentiable utility that $\mathbf{v}$-rationalizes non-invertible choice data.

**Theorem 4.** *If $\mathbf{v} \ll \mathbf{1}$ and $\mathcal{D}$ satisfies* $\text{SARP}_{\mathbf{v}}$*, then it is strongly $\mathbf{v}$-rationalizable by a strictly increasing, strictly concave, and infinitely differentiable utility function.*

Following the results in Section 3, it is easy to see that a version of Theorem 2 applies to a differentiable utility; this is, if $\mathbf{v} < \mathbf{1}$ and $\mathcal{D}$ satisfies $\text{GARP}_{\mathbf{v}}$, there is $\mathbf{v}^\star \ll \mathbf{1}$ such that $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{v}^\star}$ and $f(\mathbf{v}^\star) = f(\mathbf{v})$.

The proof of Theorem 4 uses a modified version of the Afriat inequalities.

**Lemma 7.** *If $\mathcal{D}$ satisfies* $\text{SARP}_{\mathbf{v}}$ *and $\mathbf{v} \ll \mathbf{1}$ then there are numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ for $i \in [N]$ such that for all $i, j \in [N]$*

$$u^i > u^j + \lambda^i (v_i - p^i x^j) \tag{F6}$$

*Proof.* By Lemma 2 there are numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ such that (F6) holds whenever $x^i \neq x^j$ and $u^i = u^j$ whenever $x^i = x^j$. As $v_i < 1$, whenever $x^i = x^j$ we have $p^i x^j = p^i x^i = 1 > v_i$. Therefore, as $\lambda^i > 0$, $u^i = u^j > u^j + \lambda^i (v_i - p^i x^j)$ whenever $x^i = x^j$. □

*Proof of Theorem 4.* By Lemma 7 there is $\varepsilon > 0$ small enough such that

$$u^i - \varepsilon g(x^i - x^j) > u^j + \lambda^i (v_i - p^i x^j) \qquad \text{for all } i, j \in [N] \tag{F7}$$

$$\lambda^i p^i_k > \varepsilon \qquad \text{for all } i \in [N],\ k \in [K] \tag{F8}$$

[17] In a strict sense, equality between the marginal rate of substitution and the price ratio is necessary only when goods are consumed in a strictly positive amount. See Ugarte (2023b) for more details.

Define the function

$$U(x) = \min_{i\in[N]} u^i - \lambda^i(v_i - p^i x) - \varepsilon g(x - x^i)\,.$$

Since each function $u^i - \lambda^i(v_i - p^i x) - \varepsilon g(x - x^i)$ is continuous, strictly concave, and strictly increasing,[18] $U(x)$ inherits these properties. Furthermore, from (F7) we have

$$U(x^i) = \min_{j\in[J]} u^j - \lambda^j(v_j - p^j x^i) - \varepsilon g(x^j - x^i) > u^i\,.$$

Although $U$ is continuous, strictly increasing, and strictly concave, it is not differentiable. Following Chiappori and Rochet (1987) we smooth $U$ by convolution. For $\delta > 0$ denote by $B(\delta)$ the ball or radius $\delta$ centered at $\mathbf{0}$. As $U(x)$ is continuous and $U(x^i) > u^i$, there is $\eta$ small enough such that for all $i \in [N]$ whenever $x \in B(\eta)$ we have $U(x^i - x) > u^i$. Define

$$\rho_1(x) = \begin{cases} \left[\int_{\mathbb{R}^K} \exp\left(-\frac{1}{||y||^2-1}\right) dy\right]^{-1} \exp\left(-\frac{1}{||x||^2-1}\right) & \text{if } ||x|| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\rho(x) = \frac{1}{\eta}\rho\left(\frac{x}{\eta}\right)$$

So $\rho(x)$ is continuous, infinitely differentiable,[19] symmetric ($\rho(x) = \rho(-x)$), weakly positive ($\rho(x) \ge 0$), and strictly positive whenever $||x|| < \eta$. It also satisfies

$$\int_{B(\mathbf{0},\eta)} \rho(\xi)\, d\xi = 1, \text{ and} \tag{F9}$$

[18] $\phi^i$ is strictly increasing since from (F8) for all $k \in [K]$ we have

$$\frac{\partial \phi^i(x)}{\partial x_k} = \lambda^i p^i_k - \varepsilon\left(\frac{(x_k)^2}{M + \sum_{k\in[K]}(x_k)^2}\right) > \lambda^i p^i_k - \varepsilon > 0\,.$$

[19] In particular all the partial derivatives on $\rho(x)$ of any order equal zero whenever $||x|| = \eta$.

$$\int_{B(\mathbf{0},\eta)} \xi\rho(\xi)\, d\xi = 0\,. \tag{F10}$$

Define

$$\begin{aligned}
V(x) &= (U \star \rho)(x) \\
&= \int_{\mathbb{R}^K} U(x-\xi)\rho(\xi)\, d\xi \\
&= \int_{B(\eta)} U(x-\xi)\rho(\xi)\, d\xi
\end{aligned} \tag{F11}$$

$V$ is infinitely differentiable, strictly concave, and strictly increasing (see Chiappori & Rochet, 1987). Finally take $x$ satisfying $p^i\, x \le v_i$. Then

$$\begin{aligned}
V(x) &= \int_{B(\eta)} U(x-\xi)\rho(\xi)\, d\xi \\
&= \int_{B(\eta)} \left[\min_{j\in[N]} u^j - \lambda^j(v_j - p^j\,(x-\xi)) - \varepsilon g(x-\xi-x^j)\right]\rho(\xi)\, d\xi \\
&\le \int_{B(\eta)} \left[u^i - \lambda^i(v_i - p^i\,(x-\xi)) - \varepsilon g(x-\xi-x^i)\right]\rho(\xi)\, d\xi \\
&= \left[u^i - \lambda^i(v_i - p^i\, x)\right]\int_{B(\eta)} \rho(\xi)\, d\xi - \lambda^i p^i \int_{B(\eta)} \xi\rho(\xi)\, d\xi - \\
&\qquad\qquad - \varepsilon \int_{B(\eta)} g(x-\xi-x^i)\rho(\xi)\, d\xi \\
&= u^i - \lambda^i(v_i - p^i x) - \varepsilon \int_{B(\eta)} g(x-\xi-x^i)\rho(\xi)\, d\xi \\
&< u^i - \lambda^i(v_i - p^i x) \\
&\le u^i \\
&= \int_{B(\eta)} u^i\rho(\xi)\, d\xi \\
&< \int_{B(\eta)} U(x^i-\xi)\rho(\xi)\, d\xi \\
&= V(x^i)\,.
\end{aligned}$$

The second line follows from the definition of $U$; the third one from $i \in [N]$; the fourth one rearranges terms; the fifth one from (F9) and (F10); the sixth one from the positivity of $g$; the seventh one from $\lambda^i > 0$ and $p^i x \leq v_i$; the eight one from (F9); the ninth one from $u^i < U(x^i - \xi)$ whenever $\xi \in B(\eta)$; and the tenth one from the definition of $V$. We conclude that $V$ strongly **v**-rationalizes $\mathcal{D}$. □

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