Section 1 : Strategic Games

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These notes are based in the corresponding chapters of Martin Osborne and Ariel Rubinstein's book "A Course in Game Theory", which I refer to as OR. Some ideas were also taken from Martin Osborne's book "An introduction to Game Theory", which I refer to as O.

Summary

This section starts with the study of the simpler situation of agents strategic interaction: strategic games (a.k.a. normal form games). We introduce the main ideas and present the basic equilibrium concept: Nash Equilibrium. After that we study two refinements of the idea of Nash equilibrium: trembling hand perfection and evolutionary stability.

1.1 Basic Notions

A strategic game in a setting in which a set of players make choices all the same time, and each player has preferences not only about her choice but about the *profile of choices*.

Definition 1.1. A **strategic game** is a tuple $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ where

- N is a set of players;
- A_i is the set of Player i's possible actions (we denote the set of action profiles $A = \times_{i\in N} A_i$); and
- $u_i : A \to \mathbb{R}$ is Player i's utility function, which depends on the action profile $a \in A$.

When we think about what players can do, we allow for them to *randomize*. This is, instead of choosing an action $a_i \in A_i$, Player i's *strategy* is a probability measure on A_i, which we denote $\alpha_i \in \Delta A_i$. We assume that each utility function u_i can be extended from A to ΔA by taking the expected value. This is, for a profile of strategies $\alpha \in \times_{i\in N}\Delta A_i$ we have that

$$U_i(\alpha) = \sum_{a\in A}\left(\prod_{j\in N}\alpha_j(a_j)\right)u_i(a)\,.$$

A basic requirement is for players to act optimally according to their preferences (represented by their utility function). As each player's utility depend on the whole profile of strategies, her optimal strategy will depend on what the other players are doing. This leads to the idea of a *best response*. For this, denote by α_{-i} the strategy profile of the other players; this is, $\alpha_{-i} \in \times_{j\in N\setminus\{i\}}\Delta A_j$.

Definition 1.2. Player i's **best response correspondence** $BR_i : \times_{j\in N\setminus\{i\}}\Delta A_j \rightrightarrows \Delta A_i$ is defined by

$$BR_i(\alpha_{-i}) = \arg\max_{\alpha_i\in\Delta A_i} U_i(\alpha_i, \alpha_{-i})$$

Our first notion to predict behavior is the notion of *Nash Equilibrium* (NE). Intuitively,a profile of strategies α is a NE if no Player can increase her utility without the other players changing their behavior, i.e., if all players are acting according to their best response.

Definition 1.3. A strategy profile $\alpha^\star$ is a **Nash Equilibrium** (NE) if for every $i \in N$ and $\alpha_i \in \Delta A_i$ we have

$$U_i(\alpha^\star) \geq U_i(\alpha_i, \alpha^\star_{-i})\,.$$

Exercise 1.4. Suppose α is a NE of the game G, and for Player i there are two actions $a_i', a_i'' \in A_i$, where $a_i' \neq a_i''$, such that $\alpha_i(a_i') > 0$ and $\alpha_i(a_i'') > 0$. Show that $U_i(\delta_{a_i'}, \alpha_{-i}) = U_i(\delta_{a_i''}, \alpha_{-i})$, where $\delta_{a_i'}$ and $\delta_{a_i''}$ are the probability measures assigning probability one to a_i' and a_i'', respectively. (This is, show that for a player to randomize between two actions in equilibrium she has to be indifferent between the two.)

Solution. To simplify notation, for each action $a_i \in A_i$ write $U_i(a_i) = U_i(\delta_{a_i}, \alpha_{-i})$. We have that

$$U_i(\alpha) = \sum_{a_i\in A_i} \alpha_i(a_i) \sum_{a_{-i}\in A_{-i}} \left(\prod_{\substack{j\in N\\ j\neq i}} \alpha_j(a_j) \right) u_i(a_i, a_{-i}) = \sum_{a_i\in A_i} \alpha_i(a_i) U_i(a_i)\,.$$

Define the strategy α_i' for Player i as follows

$$\alpha_i'(a_i) = \begin{cases} \alpha_i(a_i') + \alpha(a_i'') & \text{if } a_i = a_i' \\ 0 & \text{if } a_i = a_i'' \\ \alpha_i(a_i) & \text{otherwise.} \end{cases}$$

We have that

$$U_i(\alpha_i', \alpha_{-i}) = \sum_{a_i\in A_i} \alpha_i'(a_i) U_i(a_i) = \sum_{\substack{a_i\in A_i\\ a_i\neq a_i', a_i''}} \alpha_i(a_i) U_i(a_i) + (\alpha_i(a_i') + \alpha_i(a_i''))U_i(a_i')\,.$$

Hence

$$U_i(\alpha) - U_i(\alpha') = \alpha_i(a_i'')(U_i(a_i'') - U_i(a_i')) < 0\,.$$

Therefore α'_i is a profitable deviation for i, and α is not a NE. $\square$

The seminal result regarding strategic games is Nash's (1950)[1] proof that an equilibrium exists.

Theorem 1.5 (Nash, 1950). *Every game with a finite number of players and action profiles has at least one Nash equilibrium.*

The proof is basically an elegant application of Kakutani's fixed point theorem.[2] I encourage the interested reader to read the original paper (it is only 1 page long!).

1.2 Evolutionary Stability

Evolutionary stability is a criteria to analyze population dynamics. The motivation is the following: suppose we have a large population of individuals who are all the same, and each one can choose an action $a \in A$. This individuals are randomly matched in pairs, and play a completely symmetric game $G = \langle 1, 2, (A, A), (u_1, u_2) \rangle$. We interpret payoffs being related to probability of survival and reproduction. The question we want to ask is *which actions seem reasonable to see in the long run.*

Suppose we start by a population in which all individuals play action $a^\star \in A$, and a small share ε of the population mutate and start playing a different action $a \in A$. Then, the population playing a should disappear in the long run if their payoff is lower than the payoff of the population playing $a^\star$. This is

$$(1-\varepsilon)u(a^\star, a^\star) + \varepsilon u(a^\star, a) > (1-\varepsilon)u(a, a^\star) + \varepsilon u(a, a)\,. \tag{1.1}$$

If (1.1) holds for any $a \neq a^\star$, then we should expect $a^\star$ to be the action that we see in the long run, and we call this action an *evolutionary stable strategy.*

Definition 1.6. The action $a^\star \in A$ is an **evolutionary stable strategy** (ESS) of the symmetric game G if and only if there is $\varepsilon > 0$ such that (1.1) holds for any $a \neq a^\star$.

The following lemma gives us a nice criteria for finding ESSs.

Lemma 1.7. *The action $a^\star \in A$ is an ESS if and only if for any $a \neq a^\star$ either*

1. *$u(a^\star, a^\star) > u(a, a^\star)$; or*

[1] Nash, John (1950). Equilibrium Points in n-Person Games. *Proceedings of the National Academy of Sciences of the United States of America, 36*, 48-49.

[2] Kakutani, Shizuo (1941). A generalization of Brouwer's fixed point theorem. *Duke Mathematical Journal, 8*, 457-459.

2. *$u(a^\star, a^\star) = u(a, a^\star)$ and $u(a^\star, a) > u(a, a)$.*

The next one gives us some properties of ESSs.

Lemma 1.8. 1. *If $a^\star$ is an ESS then $(a^\star, a^\star)$ is a NE.*

2. *If $(a^\star, a^\star)$ is a strict NE ($u(a^\star, a^\star) > u(a^\star, a)$ for all a) then $a^\star$ is an ESS.*
3. *If $u_i(a, a) \neq u_i(a', a')$ for any $a, a' \in A$, then there is a mixed strategy α that is an ESS.*

Note in particular the counterpositive of the first result: If (a, a) is not a NE, then a *is not* an ESS.

1.3 Rationalizability and Dominance

The idea behind the concept of NE is that each player knows the strategies the other players are playing. However, in some cases this assumption may be too strong. Rationalizability and dominance are two concepts that allow us to make predictions if we want to relax the previous assumption.

Definition 1.9. An action $a_i \in A_i$ is **rationalizable** if for each $j \in N$ there exist $Z_j \subseteq A_j$ such that (1) $a_i \in Z_i$, and (2) every action $a_j \in Z_j$ is the best response to some belief of player j that assigns strictly positive probability only to other players' actions $a_j \in Z_{-j}$.

The following exercise helps is an application of the idea of rationalizability.

Exercise 1.10 (OR 56.4)**.** *(Cournot Duopoly)* Consider the strategic game $\langle \{1, 2\}, (A_i), (u_i) \rangle$ in which $A_i = [0, 1]$ and $u_i(a) = a_i(1 - a_1 - a_2)$ for $i = 1, 2$. Show that each player's only rationalizable action is his unique NE action.

Solution. First, note that Player i's best response is $a_i^{BR}(a_{-i}) = {}^{(1-a_{-i})}\!/_2$. Thus the NE is $a^\star = ({}^1\!/_3, {}^1\!/_3)$.

Let Z_i be the set of rationalizable actions, with $m = \inf Z_i$ and $M = \sup Z_i$. As the game is completely symmetric, we have that $Z_1 = Z_2 = Z$. Take a belief μ of Player i about Player $-i$'s actions, which has expected value $a_\mu \equiv \mathbb{E}_\mu[a_{-i}]$. Player i's best response given this belief is $a_i^{BR}(\mu) = {}^{(1-a_\mu)}\!/_2$.

As supp $\mu \subseteq [m, M]$, we have that $a_\mu \in [m, M]$, and therefore $a_i^{BR}(\mu) \in [{}^{(1-M)}\!/_2, {}^{(1-m)}\!/_2]$. By definition, if $a_i \in Z$ then it is a best response to some belief μ, so $m \geq {}^{(1-M)}\!/_2$ and $M \leq {}^{(1-m)}\!/_2$. These two inequalities and the fact that $m \leq M$ imply that $m = M = {}^1\!/_3$. $\square$

	b_1	b_2	b_3	b_4
a_1	0,7	2,5	7,0	0,1
a_2	5,2	3,3	5,2	0,1
a_3	7,0	2,5	0,7	0,1
a_4	0,0	0,-2	0,0	10,-1

Figure 1.1: The game in Exercise 1.11.

Exercise 1.11. Find the set of rationalizable actions of each player in the two-player game in Figure 1.1.

Solution. The actions of Player 1 that are rationalizable are a_1, a_2, and a_3; those of Player 2 are b_1, b_2, and b_3. The actions a_2 and b_2 are rationalizable since (a_2, b_2) is a NE. Since a_1 is a best response to b_3, b_3 is a best response to a_3, a_3 is a best response to b_1, and b_1 is a best response to a_1 the actions a_1, a_3, b_1, and b_3 are rationalizable. The action b_4 is not rationalizable since if the probability that Player 2's belief assigns to a_4 exceeds $1/2$ then b_3 yields a payoff higher than does b_4, while if this probability is at most $1/2$ then b_2 yields a payoff higher than does b_4. The action a_4 is not rationalizable since without b_4 in the support of Player 1's belief a_4 is dominated by a_2.

That b_4 is not rationalizable also follows from it being strictly dominated by the mixed strategy that assigns the probability $1/3$ to b_1, b_2, and b_3. □

The second approach to the question about what is reasonable for player to play if they do not know other players' strategies is the idea of eliminating *strictly dominated actions.*

Definition 1.12. The action $a_i \in A_i$ is **strictly dominated** if there is a mixed strategy α_i of Player i such that $u_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$.

The next result shows the equivalence between rationalizability and strategies that are not strictly dominated.

Proposition 1.13. *If $X \subseteq A$ survives iterated elimination of strictly dominated actions, then for every $i \in N$ the set of Player i's rationalizable actions is X_i.*

1.4 Trembling Hand Perfection

The idea behind trembling hand perfection is to allow players to make small (uncorrelated) mistakes when they choose. The idea is the following. Suppose Player i wants to choose action a_i, i.e. wants to choose the strategy σ_i with $\sigma_i(a_i) = 1$. However, instead she chooses a strategy σ_i' which is completely mixed but still gives high probability to a_i (we interpret the small probabilities of the other strategies as mistakes). Now suppose the same is true for all the players, so they want to play the strategy profile $\sigma = (\sigma_i)_{i\in N}$ but instead play $\sigma' = (\sigma'_i)_{i\in N}$. The question we want to ask is the following. Given the other players' mistakes σ'_{-i}, soed Player i still wants to play σ_i? If a strategy profile satisfies this property we say that it is a *trembling hand perfect equilibrium.*

Definition 1.14. The strategy profile σ is a **trembling hand perfect equilibrium** of a finite strategic game if and only if there exists a sequence $(\sigma^k)_{k=1}^{\infty}$ of completely mixed strategy profiles such that for every $i \in N$ the strategy σ_i is a best response to σ^k_{-i} for all values of k.

We finish this section with a couple of exercises

Exercise 1.15 (2016 Midterm)**.** Consider the following Matching Pennies game in which Player 1 has an outside option $x \in (0, 1)$ shown in Figure 1.2. Find the set of mixed strategies

	H	T
$O(ut)$	$x, 0$	$x, 0$
H	1,-1	-1,1
T	-1,1	1,-1

Figure 1.2: The game in Exercise 1.15.

for Player 1 that survive iterated elimination of strictly dominated actions. Are these strategies rationalizable? Find the sets of all NE and Trembling Hand Perfect equilibria.

Solution. Let Player 1's strategy be $(p_1, p_2, 1-p_1-p_2)$, and Player 2's strategy be $(q, 1-q)$. Then

$$u_1 = p_1x+p_2(q-(1-q))+(1-p_1-p_2)(-q+(1-q)) = p_1x+p_2(2q-1)+(1-p_1-p_2)(1-2q)\,.$$

If $2q = 1$ then it is optimal for Player 1 to set $p_1 = 1$ ($p_2 = 0$, $1-p_1-p_2 = 0$); if $2q > 1$ then it is optimal for Player 1 to set $1 - p_1 - p_2 = 0$, and if $2q < 1$ then it is optimal for Player 1 to set $p_2 = 0$. Therefore a strategy in which Player 1 plays both H and T with strictly positive probability is strictly dominated. As Player 2's strategy now depends if Player 1 plays H or T with positive probability, we cannot eliminate any Player 2's strategy (we cannot eliminate mixing since is possible when $p_1 = 1$). Furthermore, we cannot eliminate any other strategy. The set of rationalizable strategies is the same as the set of strategies that survive iterated elimination of strictly dominated actions.

There is no pure strategy NE. To allow Player 2 to mix we need

$$U_2(H) = U_2(T) \iff -p_2 + (1 - p_1 - p_2) = p_2 - (1 - p_1 - p_2) \iff 1 - p_1 - p_2 = p_2\,.$$

Since from the first part we know that in any NE either $p_2 = 0$ or $1 - p_1 - p_2 = 0$, from the previous condition we get that in any NE Player 1's strategy is $(1, 0, 0)$. For this strategy to be optimal we need

$$U_1(O) \geq U_1(H) \iff x \geq q - (1-q) \iff q \leq \frac{1+x}{2}$$
$$U_1(O) \geq U_1(T) \iff x \geq -q + (1-q) \iff q \geq \frac{1-x}{2}\,.$$

Therefore, any strategy profile $((1, 0, 0), (q, 1-q))$ with $q \in [{}^{1-x}/_2, {}^{1+x}/_2]$ is a NE.

All these equilibria are Trembling Hand perfect. Let $\sigma_1 = (1, 0, 0)$ and $\sigma_2 = (q, 1-q)$ with $q \in [{}^{1-x}/_2, {}^{1+x}/_2]$. As σ_2 is completely mixed, take $\sigma_2^k = \sigma_2$ for all $k \in \mathbb{N}$. Finally, take the sequence $(\varepsilon^k)_{k\in\mathbb{N}}$ with $\varepsilon^k > 0$, and let the candidate for Player 1's strategy be

$$\sigma_1^k = (1 - \varepsilon^k, z\varepsilon^k, (1-z)\varepsilon^k) \to \sigma_1\,.$$

For σ_2 to be a best response to σ_1^k we need

$$U_2(\sigma_1^k, H) = U_2(\sigma_1^k, T) \iff z\varepsilon^k - (1-z)\varepsilon^k = -z\varepsilon^k + (1-z)\varepsilon^k \iff z = \frac{1}{2}\,.$$

So σ_2 is a best response to σ_1^k when $z = {}^1/_2$, and we found the required sequence. □

Exercise 1.16. Consider the variant of the Hawk-Dove game shown in Figure 1.3. (when

	D	H
D	1,1	0,2
H	2,0	$1-c, 1-c$

Figure 1.3: The game in Exercise 1.16.

$c > 1$ the game has the standard Hawk-Dove structure). Find of all Nash and trembling hand perfect equilibria for all values of c. Are the equilibrium strategies evolutionary stable?

Solution. It is clear that the value of c is relevant for a player when the other player is playign H. Moreover, H is best response if and only if $c \leq 1$. We analyze the three cases

1. $c < 1$. In this case H is strictly dominant, so (H, H) is the unique NE, which is trembling hand perfect, and H is an evolutionary stable strategy.
2. $c > 1$. In this case there are two pure strategy NE, (D, H) and (H, D) and a mixed strategy NE in which both players play D with probability ${}^{(c-1)}/_c$. The mixed strategy NE is clearly trembling hand perfect. Consider the sequence of strategies $(1-\varepsilon^k, \varepsilon^k) \to (1, 0)$. The action H is a best response to this each element of the sequence if

$$2(1-\varepsilon^k) + (1-c)\varepsilon^k \geq 1 - \varepsilon^k$$

which is clearly true if ε^k is small enough. Finally, take the sequence of strategies $(\varepsilon^k, 1-\varepsilon^k) \to (0,1)$. The action D is a best response to each element of the sequence if

$$\varepsilon^k \geq 2\varepsilon + (1-c)(1-\varepsilon)$$

which again is true for ε^k small enough. Therefore the NE (D,H) and (H,D) are trembling hand perfect.

Our only candidate for a ESS is the mixed strategy $\alpha = ((c-1)/c, 1/c)$. As this is a completely mixed strategy equilibrium, it is clear that $u(\alpha,\alpha) = U(D,\alpha) = U(H,\alpha)$. From Lemma 1.7 we have that α is an ESS if and only if $U(\alpha, D) > U(D,D)$ and $U(\alpha, H) > U(H,H)$. However $U(\alpha, D) = (c-1)/c < 1 = U(D,D)$, so α is not an ESS.

3. $c = 1$: There are three NE: (D,H), (H,D), and (H,H). We check trembling hand perfection:
   - (H,H). As H is strictly better than D for Player i when Player $-i$ plays D, then adding tremble towards D for Player $-i$ does not make playing D for Player i more attractive. Therefore (H,H) is trembling hand perfect.
   - (D,H) (and (H,D)). Take the sequence of strategies $(\varepsilon^k, 1-\varepsilon^k) \to (0,1)$. The action D is a best response to each element of the sequence if $\varepsilon^k \geq 2\varepsilon$, which is obviously not true, so (D,H) and (H,D) are not trembling hand perfect.

   Finally, our only candidate for an ESS is H. Note that $U(H,H) = U(D,H)$ and $U(H,D) > U(D,D)$, so H is an ESS.

$\square$

Section 2 : Extensive Games with Perfect and Imperfect Information

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These notes are based in the corresponding chapters of Martin Osborne and Ariel Rubinstein's book "A Course in Game Theory", which I refer to as OR. Some ideas were also taken from Martin Osborne's book "An introduction to Game Theory", which I refer to as O.

Summary

In this section we study extensive games. We first revise the setup of the problem in order to get familiar with the notation. Then we study equilibria, first in games with perfect information and then in games with imperfect information (when players may not have all the information about what happened in the past). The main takeaway is that the concept of Nash equilibrium seems to be insufficient in these setting. Therefore, we study stringer equilibrium concepts, which add additional requirements. These are subgame perfect equilibrium in the case with perfect information, and sequential equilibrium in the case with imperfect information. At the end we focus on a specific case of extensive games of imperfect information, known as Bayesian extensive games.

2.1 Extensive Games of Perfect Information

2.1.1 Setup

According to OR, an **extensive game** is an "explicit description of the *sequential structure* of the decision problems encountered by players in specific situations". The main characteristic of this structure is that players make decisions not only at the beginning of the game but whenever have to play. Therefore, they can revise their "original" plan of action. In the specific case of perfect information, at each point where a player has to choose, she knows *everything* that has happened before that point.

Definition 2.1. An **finite extensive game with perfect information** Γ is a tuple $\Gamma = \langle N, H, P, (\succsim_i)_{i \in N} \rangle$, where

- N is the set of players.
- H is a set of finite sequences, called **histories**. Each history $h = (a^k)_{k=1,\dots,K}$ describes the first K actions taken in the game. The set H satisfies the following properties
  - $\emptyset \in H$; and
  - if $(a^k)_{k=1,\dots,K} \in H$ and $L < K$, then $(a^k)_{k=1,\dots,L} \in H$.

  We say that a history $(a^k)_{k=1,\dots,K} \in H$ is **terminal** if there is no action a^{K+1} such that $(a^k)_{k=1,\dots,K+1} \in H$, and call Z the set of terminal histories. A history is **nonterminal** if it is not terminal.
- $P : H \setminus Z \to N$ is the **player function**, which assigns to each nonterminal history (each member of $H\ Z$) a player (a member of N). $P(h)$ is the player who has to take an action after history h.
- For each player $i \in N$, $\succsim_i$ is her preference relation on the set of terminal histories Z.

Generally, finite extensive games with perfect information can be represented (if they are simple enough) graphically by a rooted tree whose leafs are payments and all the other vertices are players, while its edges are actions.[1] The following exercise helps us to relate both representations.

Exercise 2.2. Formally define all the components of the game shown in Figure 2.1. If it's easier use utility functions instead of preference relations.

[1]Formally, a rooted tree is a specific type of graph, and a graph G is a pair (V, E), where V is a set of vertices and $E \subseteq V \times V$ is a set of directed edges, that is, if $(a, b) \in E$, then (a, b) is interpreted as an edge going from vertex a to vertex b. A path p in G is a sequence $v_1, \dots, v_n$ of edges in G such that for every $i \in \{1, \dots, n-1\}$ we have $(v_i, v_{i+1}) \in E$. The path p is said to go from a to b if $v_1 = a$ and $v_n = b$ in the previous definition, that is, if the starting vertex of p is a and the ending vertex of this path is b. A graph is said to be a rooted tree if there exists a unique vertex $r \in V$ (called the root) such that for every vertex $a \in V$ with $a \neq r$, there exists a unique path from r to a. Finally, a vertex b is a *leaf* of a rooted tree if there is no edge of the form (b, b') for any vertex b'.

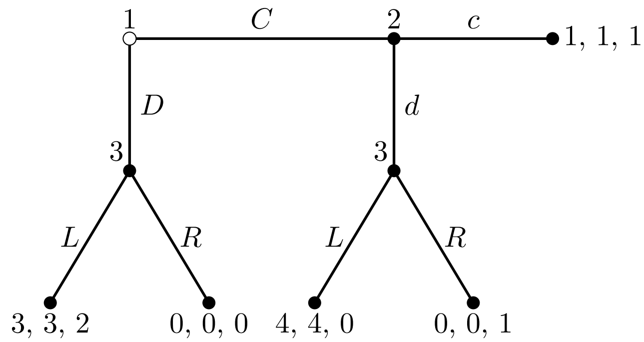


Figure 2.1: A perfect information version of Selten's Horse.

2.1.2 Strategies, Outcomes, and Nash Equilibrium

In our definition of a strategy we need to include the fact that players may revise their action plans whenever they have to play. To do this, for each nonterminal history h we denote by $A(h)$ the set of feasible actions for Player $P(h)$; this is $A(h) = \{a : (h, a) \in H\}$. We use this set to define strategies.

Definition 2.3. For $i \in N$, a **strategy** of Player i in the game Γ is a function s_i that assigns an action $a \in A(h)$ to every nonterminal history $h \in H \setminus Z$ for which $P(h) = i$. A **strategy profile** is a collection of strategies $s = (s_i)_{i \in N}$

Intuitively, a strategy tells us what each player will do at every point she has to play. If Player i follows the strategy s_i, the for every history h with $P(h) = i$ she will choose $s_i(h)$.

Note that each strategy profile s leads to an **outcome** $O(s)$, which is the terminal history that results when each player $i \in N$ follows the strategy s_i.[2] Given what we know from the study of strategic games, it seems natural to extend the solution concept of Nash equilibrium to this setting.

Definition 2.4. A **Nash equilibrium** (NE) of Γ is a strategy profile $s^\star$ such that for every player $i \in N$ and strategy s_i we have that

$$O(s^\star) \succsim_i O(s^\star_{-i}, s_i)$$

[2] This is, $O(s)$ is the terminal history $(a^k)_{k=1,\ldots,K}$ satisfying $s_{P((a^j)_{j=1,\ldots,k})}((a^j)_{j=1,\ldots,k}) = a^{k+1}$ for every $k = 0, \ldots, K-1$.

A simple way of finding the set of Nash equilibria of a game Γ is to consider its *strategic form*. This is, to think of Γ as a strategic game where instead of actions players choose strategies. Formally the definition is

Definition 2.5. The **strategic form** of Γ is the strategic game $\langle N, (S_i)_{i \in N}, (\succsim'_i)_{i \in N} \rangle$, where for each player $i \in N$

- S_i is the set of Player i's strategies in Γ;
- $\succsim'_i$ is defined by $s \succsim'_i s' \iff O(s) \succsim_i O(s')$ for all $s, s' \in \times_{i \in N} S_i$.

We use the strategic form to find the Nash equilibria in the following exercise.

Exercise 2.6. Find the pure strategy Nash equilibria on the game shown in Figure 2.1.

2.1.3 Subgames and Subgame Perfect Equilibrium

The concept of NE has two particular properties in extensive games:

- It does not require to define a strategy profile completely. In particular, it does not need to define a player's actions in histories that are inconsistent with her previous actions; and
- Can lead to unreasonable actions off the equilibrium path, which themselves define the equilibrium path.

Exercise 2.7. Find the pure strategy Nash equilibria of the game shown in Figure 2.2.

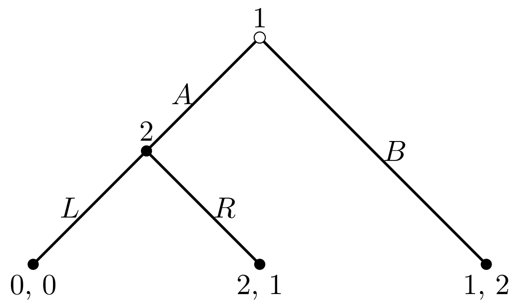


Figure 2.2: (OR 96.2) A two player extensive game.

While B, L is a Nash equilibrium, its seems to be naive from Player 1's perspective.

- Player 1 chooses B because if not then Player 2 will choose L
- But Player 2's choice of L is suboptimal if Player 1 plays A

Intuitively, Player 1 is choosing B because of the threat that, if she chooses A, Player 2 will play L. However, this it seems unreasonable for Player 1 to believe Player 2's threat as if Player 1 chooses A then it is optimal for Player 2 to choose R.

To solve this problem we need to define a new equilibrium concept, in which we ask players to choose optimally in every history in which they have to play (i.e., after every h for which $P(h) = i$), independent if these histories occur in equilibrium. In order to accomplish this, we need to define the *subgames* of the game.

Definition 2.8. Given a finite extensive game with perfect information Γ and a nonterminal history $h \in H \setminus Z$, the **subgame that follows** h is the extensive game with perfect information $\Gamma(h) = \langle N, H|_h, P|_h, (\succsim_i|_h)_{i \in N} \rangle$, where

- $H|_h$ is the set of sequences h' of actions for which $(h, h') \in H$;
- $P : H|_h \to N$ is defined by $P|_h(h') = P(h, h')$; and
- $\succsim_i|_h$ is defined by $h' \succsim_i|_h h'' \iff (h, h') \succsim_i (h, h'')$.

In simple, after each nonterminal history h Players *start a new game*. This game is the remainder of the original game after the actions that already occurred. As each subgame is itself a game, we should require them to act optimally after each history. This requirement is called *subgame perfection.*

Definition 2.9. A **subgame perfect equilibrium** (SPE) of the extensive game with perfect information Γ is a strategy profile $s^\star$ for which for any nonterminal history $h \in H \setminus Z$ the strategy profile $s^\star|_h$ is a Nash equilibrium of $\Gamma(h)$ (where $s|_h$ refers to what the strategy prescribes after history h).

A couple of comments on the relation between NE and SPE:

- a SPE of the game Γ is also a NE of this game (why?);
- a NE is not necessarily a SPE (verify this in the game shown in Figure 2.2), i.e., SPE is a *refinement*.

2.1.4 Existence of subgame perfect equilibrium and the one deviation property

Exercise 2.10. Show that (when the game is extended to mixed strategies) every finite extensive game with perfect information has a NE equilibrium.

As the requirements for a subgame perfect equilibrium are stronger than the requirements for the Nash equilibrium, the existence of a subgame perfect equilibrium is not obvious. In order to prove the existence of a subgame perfect equilibrium we use the following result, known as the *one deviation property.* This result allows us to narrow our search of profitable deviations when looking for an equilibrium.

Proposition 2.11 (The one deviation property)**.** *The strategy profile $s^\star$ is a subgame perfect equilibrium of the finite horizon extensive game with perfect information Γ if and only if for every player $i \in N$ and every history $h \in H$ for which $P(h) = i$ we have*

$$O_h(s^\star|_h) \succsim_i |_h O_h(s^\star_{-i}|_h, s_i|_h)$$

for every strategy $s_i|_h$ of Player i in the subgame $\Gamma(h)$ that differs from $s^\star_i|_h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

An intuitive "proof". Here we show in an example why the one-deviation property works. The general proof (below) is just a generalization of the same argument. Start from the game in Figure 2.3 and the strategy profile specified there.

To see the equivalence between an SPE and the one-deviation property we analyze both directions.

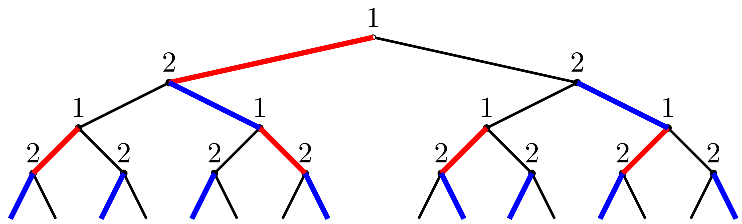


Figure 2.3: Example of a dynamic game and a strategy profile.

1. If the strategy is an SPE, then it is clear that it satisfies the one-deviation property (since there are no profitable deviations).
2. If the strategy is not an SPE then we want to show that there is a one-shot profitable deviations and hence it fails the one-deviation property. Suppose there is a profitable deviation that is not a one-shot deviation, and suppose that this deviation is optimal. For example suppose the optimal strategy for Player 2 is the one in Figure 2.4.

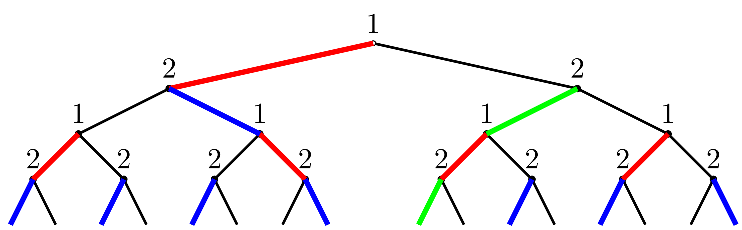


Figure 2.4: A profitable deviation.

In Figure 2.4 we see a profitable deviation in which Player 2 changed her strategy in the subgame after Player 1 played R. Specifically, Player 2 chooses L (instead of R) when the history is R and L (instead of R) when the history is RLL.
If the strategy in Figure 2.4 is the optimal strategy, that implies that to play L is optimal after the history RLL. This implies that another profitable deviation from the strategy in Figure 2.3 is the one shown in Figure 2.5.

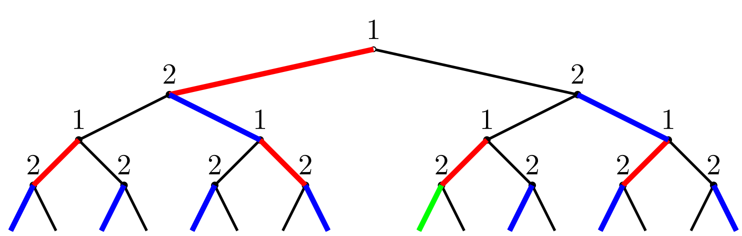


Figure 2.5: A one-shot deviation.

Note that the strategy in Figure 2.5 is both profitable and a one-shot deviation. Hence the original strategy does not satisfy the one-deviation property.

Proof of Proposition 2.11. If $s^\star$ is a subgame perfect equilibrium it is clear that it satisfies the condition. To prove the other direction, suppose $s^\star$ is not a subgame perfect equilibrium. Then there is a player i who can deviate profitably in the subgame $\Gamma(h')$. From among all the profitable deviations of Player i in $\Gamma(h')$ choose the strategy $s_i|_{h'}$ for which the number of histories h such that $s_i|_{h'} \neq s_i^\star|_{h'}(h)$ is minimal (note that as the game is finite, this number also is).

Let h^* be the longest history h of $\Gamma(h')$ for which $s_i|_{h'}(h) \neq s_i^\star|_{h'}(h)$ (i.e., h^* is the "deepest deviation point"). Then, $s_i^\star|_{(h',h^*)}$ and $s_i|_{h'}|_{h^*} = s_i|_{(h',h^*)}$ differ only in the action they prescribe after history (h', h^*). Furthermore, $s_i|_{(h',h^*)}$ is a profitable deviation, since otherwise there would be a profitable deviation after h' that differs from $s_i^\star|_{h'}$ after fewer histories than does $s_i|_{h'}$. Thus, $s_i|_{(h',h^*)}$ is a profitable deviation in $\Gamma(h', h^*)$ that differs from $s_i^\star|_{(h',h^*)}$ only in the action it prescribes after the initial history of $\Gamma(h', h^*)$. □

The one deviation property tells us that to test if a strategy profile $s^\star$ is a subgame perfect equilibrium, we only need to compare $s^\star$ with other strategy profiles that differ of $s^\star$ in only one action. This property is a very powerful tool, which can be extended to infinite games (with discount factor less than one) and games with imperfect information (see OR 227.1). It also allows us to prove the existence of a subgame perfect equilibrium by construction. This is, the proof also illustrates how to find the subgame perfect equilibrium in any specific game.

Proposition 2.12. *Every finite extensive game with perfect information Γ has a subgame perfect equilibrium.*

Proof. Let $\ell(\Gamma)$ be the length of the longest history in Γ, which we call the length of the game. Also, let $R : H \to Z$ be a function that assigns a terminal history to every history $h \in H$. We propose a specific function R which specifies a subgame perfect equilibrium of the game. To do this by we proceed by induction on the length of subgames $\ell(\Gamma(h))$.

- If $\ell(\Gamma(h)) = 0$ then h is a terminal history. Define $R(h) = h$.
- Suppose $R(h)$ is defined for all $h \in H$ with $\ell(\Gamma(h)) \leq k$, and take $h^* \in H$ such that $\ell(\Gamma(h^*)) = k+1$. Let $P(h^*) = i$. As $\ell(\Gamma(h^*)) = k+1$, we have that $\ell(\Gamma(h^*, a)) \leq k$ for all $a \in A(h^*)$. Define $s_i^\star(h^*)$ to be a $\succsim_i |_h$ maximizer of $R(h^*, a)$ over $A(h^*)$. Define $R(h^*) = R(h^*, s_i^\star)$.

By induction we have now defined a strategy profile $s^\star$ in the game Γ. Also, by construction this profile satisfies the one deviation property, and is therefore (Proposition 2.11) a subgame perfect equilibrium. □

We finish the study of perfect information with an exercise

Exercise 2.13 (O 177.2, with an extension)**.** A childs action a (a number) affects both her own private income $c(a)$ and her parents' income $p(a)$; for all values of a we have $c(a) < p(a)$. The child is selfish: she cares only about the amount of money she has. Her loving parents care both about how much money they have and how much their child has. Specifically, model the parents as a single individual whose preferences are represented by a payoff equal to the smaller of the amount of money they have and the amount of money the child has. The parents may transfer money to the child. First the child takes an action, then the parents decide how much money to transfer.

1. Show that in a subgame perfect equilibrium the child takes an action that maximizes the sum of her private income and her parents income.
2. Suppose $c(a) = 1+2a-a^2$ and $p(a) = 100+2a-a^2$ (so both payments are maximized at $a = 1$). Construct a Nash equilibrium of this game in which the child chooses $a = 2$.

2.2 Extensive Games of Imperfect Information

Extensive games of imperfect information are the extension of extensive games with perfect information to situation in which a Player, at the time she has to play, may not have all the information about what happened in the past. Specifically, the main differences are.

- There are moments in the game when no player has to play. Instead changes in the environment happen according to an exogenous probability distribution. In these moments we say that *chance* (represented by c) is the one who acts.
- Players may be imperfectly informed about some (or all) moves made in the past by other players (including c).

The fact that players are not completely informed about past behavior of other players makes the evaluation of different actions more complex. When a Player has to choose an action, she needs to take into account not only what other players will do in the future, but also what she believes they did in the past (the specific reason for this will be clear later).

2.2.1 Setup

Formally, a finite extensive game with imperfect information is defined as follows

Definition 2.14. An **finite extensive game with imperfect information** is a tuple $\Gamma = \langle N, H, P, f_c, (\mathcal{I}_i)_{i\in N}, (\succsim_i)_{i\in N}\rangle$, where

- N is the set of players.
- H is the set of histories (satisfying the same properties as in Definition 2.1).
- $P : H \setminus Z \to N \cup \{c\}$ is the player function, which also includes the chance in its range (if $P(h) = c$, then chance determines the action after history h).
- A function f_c that for any history h for which $P(h) = c$ assigns a probability measure $f_c(\cdot|h)$ on $A(h)$.
- For each Player $i \in N$ an **information partition** of $\{h \in H : P(h) = i\}$. Each element $I_i \in \mathcal{I}_i$ is a **information set**, and satisfies that $A(h) = A(h')$ whenever $h, h' \in I_i$.
- For each Player $i \in N$ a preference relation $\succsim_i$ on the set of lotteries over Z, namely ΔZ, that can be represented as the expected value of a payoff function defined on Z.

A couple of comments about the previous definition:

- Each player's information partition tells us about the histories over which the Player cannot distinguish. This is, if $h, h' \in I_i$, then when Player i has to play after both h and h' and she cannot distinguish which of the two histories (or any other history in I_i) is the one that occurred.
- We do not allow more than one player to move after any history. However, there is a sense in which an extensive game as defined above can model such a situation, as shown in Figure 2.6. Although Player 1 moves first, Player 2 has not any information about Player 1's actions when she has to move. Players' interaction is essentially the same as one in a strategic game.

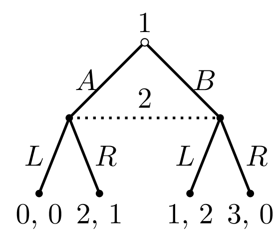


Figure 2.6: An extensive game with imperfect information and "simultaneous moves".

2.2.2 Strategies

We start from our classical definition of strategies. the main difference here is that players choose not after a specific history, but when they reach a specific information set.

Definition 2.15. Given Γ, a **pure strategy** is a function that assigns to every information set $I_i \in \mathcal{I}_i$ an action $a \in A(I_i)$.

We have two possible ways to think how a player randomizes among her possible actions

Definition 2.16. A **mixed strategy** of player i is a probability measure over the set of her pure strategies.

A **behavioral strategy** of player i is a collection $(\beta_i(I_i))_{I_i \in \mathcal{I}_i}$ of independent probability measures, where $\beta_i(I_i)$ is a probability measure over $A(I_i)$.

In games with perfect recall (as the ones we study here) there is a sense in which both mixing possibilities are equivalent. For any profile $\sigma = (\sigma_i)_{i \in N}$ of either mixed or behavioral strategies, let $O(\sigma)$ be the probability measure over terminal histories induced by σ.

Proposition 2.17. *For any mixed strategy of a player in a finite extensive game with perfect recall there is an outcome-equivalent behavioral strategy.*

Proof. OR 214.1. □

Given this equivalence, we will focus on behavioral strategies for our equilibrium definitions.

Definition 2.18. A **Nash equilibrium in behavioral strategies** of the game Γ is a profile of behavioral strategies $\beta^\star$ such that for every Player $i \in N$ and every behavioral strategy β_i of Player i we have

$$O(\beta^\star) \succsim_i O(\beta^\star_{-i}, \beta_i)\,.$$

2.2.3 Beliefs and Sequential Equilibrium

By now, it should be clear that the concept of Nash equilibrium may be insufficient.

- Each player should play optimally even in situations that do not happen (like in SPE).
- However, in this case we need for players to act optimally *at every information set.* (why?)

What does it mean for a Player to act optimally at a given information is not always straightforward to define. In the game shown in Figure 2.2.3 the idea of subgame perfection is unproblematic. It rules out the equilibrium L, r, as if for some reason Player 2 reaches her information set, she unequivocally prefers to play L instead of R.

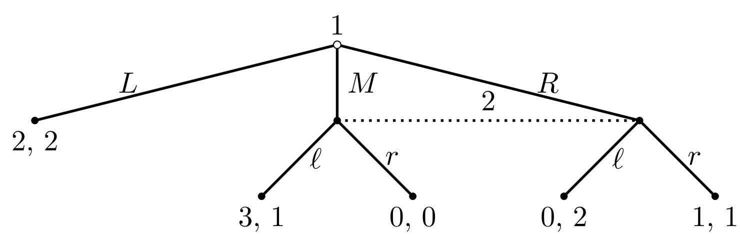


Figure 2.7: (OR 219.1) An extensive game with imperfect information in which the requirement that each players strategy be optimal at every information set eliminates a Nash equilibrium.

The game shown in 2.8 presents the (more common) situation in which subgame perfection is not a straightforward criteria.

- L, r is a Nash equilibrium;
- however, if Player 2's information set is reached her optimal action *depends on her beliefs about how she got there*:
  - if she thinks it is more likely that she reached her information set because Player 1 played M, then it is better for her to play ℓ.

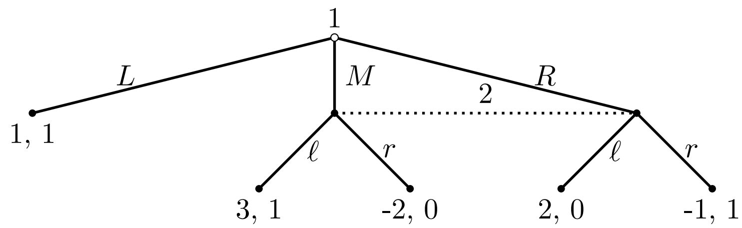


Figure 2.8: (OR 226.1) An extensive game with imperfect information that has a Nash equilibrium that cannot be ruled out simply by subgame perfection.

  - if she think it is more likely that the reason is that Player 1 played R, then r is the optimal choice.

If we want to apply the idea of subgame perfection to every extensive game with imperfect information, we need to address the problem of how players define their beliefs when they reach information sets. Therefore, instead of focusing on strategy profile we focus on *assessments*, which include both strategies *and* beliefs.

Definition 2.19. An **assessment** in the game Γ is a pair (β, μ), where β is a profile of behavioral strategies, and μ is a function that assigns to every information set a probability measure on the set of histories in that set.

Given $O(\beta)$ (the probability measure over terminal histories induced by β) and μ, a new probability measure $O(\beta, \mu | I)$ can be computed at each information set I. The analogous to subgame perfection in this setting is that whenever a Player potentially has to choose (i.e. at every information set), she is acting according to her preferences and beliefs. This requirement is called *sequential rationality.*

Definition 2.20. The assessment (β, μ) if the game Γ is **sequentially rational** if for every player $i \in N$, every information set $I_i \in \mathcal{I}_i$ and every behavioral strategy β_i' of Player i we have

$$O(\beta, \mu | I_i) \succsim_i O((\beta_{-i}, \beta_i'), \mu | I_i)\,.$$

We still have the problem about how to define beliefs.

- Natural requirement is is for beliefs to be updating using Bayes' rule.
- However, it is not always straightforward. Take the game shown in Figure 2.8 and compare the profiles M, ℓ and L, r. Both these profiles are Nash equilibria and cannot be ruled out by subgame perfection.
  - In the M, ℓ equilibrium, it is clear that Player 2's beliefs at her information set assign probability one to have reached it because Player 1 played M, since that is Player 1's strategy.

- In the L, r equilibrium the probability of Player 1 playing M or R is zero, and therefore Player 2's beliefs cannot be computed using Bayes' rule. And if we cannot compute player 2's beliefs, we cannot know if Player 2's strategy is optimal.

To overcome this problem, we use a criteria called *consistency*.

Definition 2.21. As assessment (β, μ) is **consistent** if there is a sequence $((\beta^n, \mu^n))_{n=1}^{\infty}$ of assessments that converges to (β, μ) in Euclidian space and has the properties that strategy profile β^n is completely mixed, and each belief system μ^n is derived from β^n using Bayes' rule.

Intuitively, an assessment is consistent if

- beliefs are formed using Bayes' rule when possible; and
- when this is not possible, they are the limit of the beliefs using Bayes' rule for some trembling of the behavioral strategy of the assessment.

Although the consistency requirement is rather opaque and not very intuitive, it is the criteria usually required to define beliefs in information sets that have probability zero of being reached.[3]

Sequential rationality and consistency lead to the definition of *sequential equilibirum*.

Definition 2.22. An assessment $(\beta^\star, \mu^\star)$ is a **sequential equilibrium** of the game Γ if it is sequentially rational and consistent.

It is possible to show that a sequential equilibrium exists. However, the proof is quite cumberstone. It starts by reducing the game Γ to a strategic game, for which shows that a trembling hand perfect equilibrium exists. Then, it shows that for every trembling hand perfect equilibrium of the strategic game you can construct an assessment that is a sequential equilibrium. Therefore, a sequential equilibrium must exist.

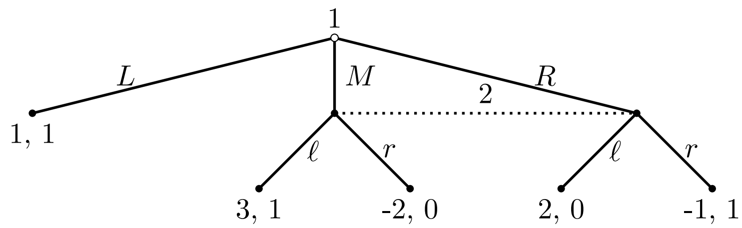
Proposition 2.23. *Every finite extensive game with imperfect information has a sequential equilibrium.*

Proof. OR Propositions 248.2, 249.1, 251.2, and Corollary 253.2. $\square$

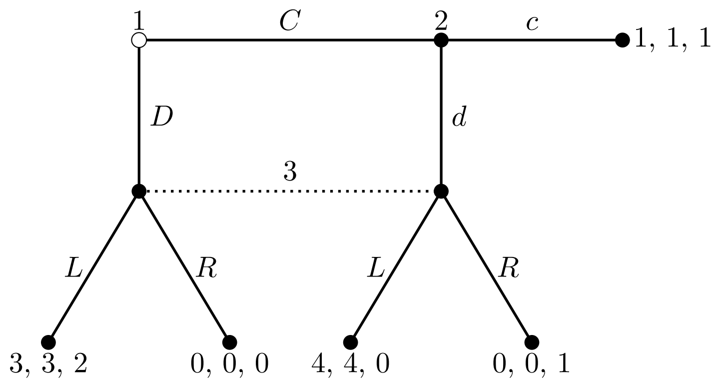
We finish the section applying the concept of sequential equilibrium to some exercises.

[3] To quote David Kreps' book *A course in Microeconomic Theory* (p. 430): "[r]ather a lot of bodies are buried in this definition".

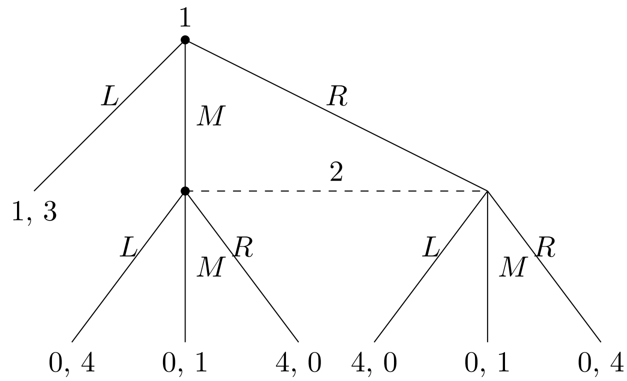
Exercise 2.24 (OR 226.1)**.** Find the set of sequential equilibria of the game in Figure 2.8.



Exercise 2.25 (OR 225.1)**.** Find the set of sequential equilibria of the Selten's Horse



Exercise 2.26 (2020 Final)**.** Find the set of sequential equilibria of the following game.



2.3 Bayesian Extensive Games

A Bayesian Game is an extensive game in which each player has a *type* which is her private information. Everything in the game is observable to every player *except* the other players' types.

2.3.1 Setup

Definition 2.27. A **Bayesian extensive game with observable actions** is a tuple $\langle \Gamma, (\Theta_i)_{i\in N}, (p_i)_{i\in N}, (u_i)_{i\in N}\rangle$ where

- $\Gamma = \langle N, H, P\rangle$ is an extensive game with perfect information and simultaneous moves (this is, $P : H \setminus Z \rightrightarrows N$).
- Θ_i is a finite set (the set of Player i's possible **types**) with a typical element $\theta_i \in \Theta_i$. We denote a profile of types by $\theta = (\theta_i)_{i\in N}$ and the set of such profiles by $\Theta = \times_{i\in N}\Theta_i$.
- $p_i \in \Delta\Theta_i$; this is, p_i is a probability measure on Θ_i. This measure has full support ($p_i(\theta_i) > 0$ for all $\theta_i \in \Theta_i$) and the measures p_i are independent.
- $u_i : \Theta \times Z \to \mathbb{R}$ is a von-Neumann Morgensten utility function.

We can think of this game as an extensive game with imperfect information (and simultaneous moves) in which in the first step chance chooses the types of the players, and then players play according to the structure of the game. The main property of the game is that the only action that Player i cannot observe is the chance's choice about Θ_{-i}. In this game the set of histories is $\{\emptyset\} \cup (\Theta \times H)$ and each information set of Player i is of the form $I_i(\theta_i, h) = \{((\theta_i, \theta_{-i}), h) : \theta_{-i} \in \Theta_{-i}\}$ for $\theta_i \in \Theta_i$ and $h \in H$.

As we can think of Bayesian extensive games with observable actions as extensive games with imperfect information, an assessment is a natural candidate for an equilibrium. However, some characteristics of the game takes us to focus in a subclass of assessments:

- As each player can observe her own type θ_i and the history of moves h, we can define a strategy for each type of each player. This is, for each Player i we need to define a family of strategies $(\sigma_i(\theta_i))_{\theta_i\in\Theta_i}$, where each strategy $\sigma_i(\theta_i)$ prescribes an action $a \in A_i(h)$ for every history $h \in H \setminus Z$ for which $i \in P(h)$.[4]
- As all players $j \neq i$ have the same prior belief about Player i's type, and all the actions are observable, they all have the same information about i. Hence, all players $j \neq i$ should have *the same* beliefs about i's type.

From the previous remarks, our equilibrium candidate is a pair (σ, μ), where $\sigma = ((\sigma_i(\theta_i))_{\theta_i\in\Theta_i})_{i\in N}$

[4] This is not a restriction with respect to the original notion of behavioral strategies, just a clear way of thinking about them in this specific setting. $A_i(h)$ is the set of actions available to Player i after history h.

is a profile of behavioral strategies and $\mu = (\mu_i(h))_{i\in N, h\in H\setminus Z}$ is a belief system. This is, for each $i \in N$ and $\theta_i \in \Theta_i$, the function $\sigma_i(\theta_i)$ is a behavioral strategy that takes a nonterminal history $h \in H \setminus Z$ such that $i \in P(h)$ and prescribes a probability measure on $A_i(h)$. And for each $i \in H$ and $h \in H \setminus Z$, $\mu_i(h)$ is a probability measure on Θ_i describing the beliefs about θ_i of each Player $j \neq i$.

2.3.2 Perfect Bayesian Equilibrium

Given a belief system μ, denote by μ_{-i} be belief system derived from μ about the types of every player except i. For any strategy profile σ, Player i's behavioral strategy s_i, and belief system μ, let $O((\sigma_{-i}, s), \mu_{-i}|h)$ be the probability measure on the set of terminal histories Z generated by the strategy profile σ_{-i} of players $j \neq i$, the belief about their types μ_{-i}, and Player i's strategy s_i. The solution concept that we use is the following

Definition 2.28. Let $\langle \Gamma, (\Theta_i)_{i\in N}, (p_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a Bayesian extensive game with observable actions. A pair $(\sigma^\star, \mu^\star) = \big((\sigma_i^\star(\theta_i))_{\theta_i\in\Theta_i, i\in N}, (\mu_i^\star(h))_{i\in N, h\in H\setminus Z}\big)$ is

- **Sequentially rational** if for every $h \in H\setminus Z$, $i \in N$, $\theta_i \in \Theta_i$, and Player i's behavioral strategy s_i we have

$$O((\sigma^\star_{-i}, \sigma^\star(\theta_i)), \mu^\star_{-i}|h) \succsim_{\theta_i} O((\sigma^\star_{-i}, s_i), \mu^\star_{-i}|h)\,,$$

  where $\succsim_{\theta_i}$ is derived from $u_i(\theta_i, \cdot)$.
- **PB-consistent** if for each $i \in N$ we have that $\mu_i^\star(\emptyset) = p_i$ and, whenever possible, $\mu_i^\star$ is derived from Bayes' rule.

The pair $(\sigma^\star, \mu^\star)$ is a **Perfect Bayesian equilibrium** (PBE) if it is sequentially rational and PB-consistent.

The question about equilibrium existence can be answered directly from our knowledge about sequential equilibrium. As we argued, every Bayesian extensive games with observable actions is an extensive game with imperfect information, where we know a sequential equilibrium exists. Moreover, the requirements of PBE are weaker than the requirements of sequential equilibrium, as PB-consistency is a weaker requirement than consistency (as it does not say anything about beliefs off the equilibrium path). Therefore, a PBE can be constructed from the each sequential equilibrium of the extensive game associated with the original game. How to do this is shown in the next proposition.

Proposition 2.29 (OR 234.1)**.** *Let (β, μ) be a sequential equilibrium of the extensive game associated with the Bayesian extensive game $\langle \Gamma, (\Theta_i)_{i\in N}, (p_i)_{i\in N}, (u_i)_{i\in N}\rangle$. For every $h \in H \setminus Z$, $i \in P(h)$, and $\theta_i \in \Theta_i$, let $\sigma_i(\theta_i)(h) = \beta_i(I(\theta_i, h))$. Then there is a collection*

$(\mu_i(h))_{i \in N, h \in H \setminus Z}$, *where* $\mu_i(h) \in \Delta\Theta_i$, *such that*

$$\mu(I(\theta_i, h))(\theta, h) = \prod_{j \in N \setminus \{i\}} \mu_j(h)(\theta_j) \quad \textit{for all } \theta \in \Theta \textit{ and } h \in H$$

and $((\sigma_i)_{i \in N}, (\mu_i)_{i \in N})$ *is a PBE of the Bayesian extensive game.*

Proof. Since (β, μ) is a sequential equilibrium there is a sequence of assessments $(\beta^n, \mu^n)_{n=1}^{\infty} \to (\beta, \mu)$. In this sequence (1) each strategy profile β^n is completely mixed and (2) each belief system μ^n is derived from β^n using Bayes' rule. For each $h \in H$, $i \in P(h)$, and $\theta_i \in \Theta_i$ let $\sigma_i^n(\theta_i)(h) = \beta_i^n(I_i(\theta_i, h))$ for each value of n. Given these (completely mixed) strategies define a profile $(\mu_i^n)_{i \in N}$ of beliefs in the Bayesian extensive game that is derived from Bayes' rule. It is straightforward to show that

$$\mu^n(I(\theta_i, h))(\theta, h) = \prod_{j \in N \setminus \{i\}} \mu_j^n(h)(\theta_j) \quad \text{for each value of } n.$$

This equality and the properties of $(\mu_i^n)_{i \in N}$ are preserved in the limit, so

$$\mu(I(\theta_i, h))(\theta, h) = \prod_{j \in N \setminus \{i\}} \mu_j(h)(\theta_j)\,.$$

Thus by the sequential rationality of the sequential equilibrium, $((\sigma_i)_{i \in N}, (\mu_i)_{i \in N})$ is sequentially rational and hence a perfect Bayesian equilibrium. □

With this result the existence of a PBE is straightforward

Proposition 2.30. *Every Bayesian extensive game with observable actions has a PBE.*

Proof. Follows directly from the previous result and the existence of a sequential equilibrium in extensive games. □

We finish the review of Bayesian extensive games by solving an exercise.

Exercise 2.31 (OR 246.2)**.** *Pre-trial negotiation* Player 1 is involved in an accident with Player 2. Player 1 knows whether she is negligent or not, but Player 2 does not know; if the case comes to court the judge learns the truth. Player 1 sends a "take-it-or-leave-it" pre-trial offer of compensation that must be either 3 or 5, which Player 2 either accepts or rejects. If she accepts the offer the parties do not go to court. If she rejects it the parties go to court and Player 1 has to pay 5 to player 2 if he is negligent, and 0 otherwise; in either case Player 1 has to pay the court expenses of 6. Formulate this situation as a signaling game and find its sequential equilibria. Suggest a criterion for ruling out unreasonable equilibria.

Section 2 : Extensive Games with Perfect and Imperfect Information

Suggested Solutions

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Exercise 2.2 Formally define all the components of the game shown in Figure 2.1. If it's easier use utility functions instead of preference relations.

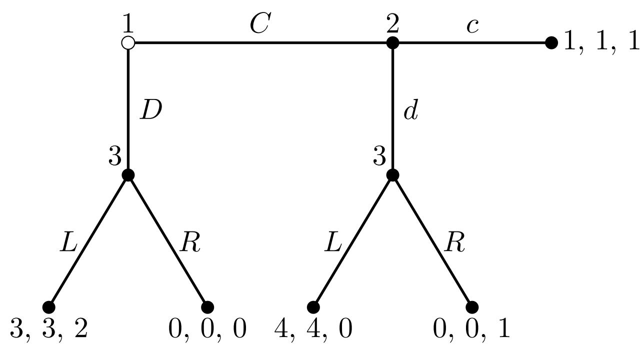


Figure 2.1: A perfect information version of Selten's Horse.

Solution.

- $N = \{1, 2, 3\}$
- $H = \{\emptyset, C, Cc, Cd, CdL, CdR, D, DL, DR\}$, $H = \{Cc, CdL, CdR, DL, DR\}$
- $P(\emptyset) = 1$, $P(C) = 2$, $P(Cd) = P(D) = 3$
- Utility functions are defined in the following table

	u_1	u_2	u_3
Cc	1	1	1
CdL	4	4	0
CdR	0	0	1
DL	3	3	2
DR	0	0	0

$\square$

Exercise 2.6 Find the pure strategy Nash equilibria on the game shown in Figure 2.1.

Solution. Denote Player 3 strategy by AB, where A is the action she plays after the history Cd, and B is the action she plays after D. The strategic form of the game is (best responses highlighted)

P1 plays C

P2 \ P3	LL	LR	RL	RR
c	1,1,1	1,1,1	1,1,1	1,1,1
d	4,4,0	0,0,1	4,4,0	0,0,1

P1 plays D

P2 \ P3	LL	LR	RL	RR
c	3,3,2	0,0,0	3,3,2	0,0,0
d	3,3,2	0,0,0	3,3,2	0,0,0

The Nash Equilibria are C, c, LR, C, c, RR, D, c, LL and D, c, RL. □

Exercise 2.7 Find the pure strategy Nash equilibria of the game shown in Figure 2.2.

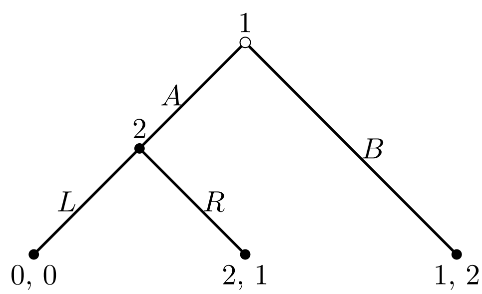


Figure 2.2: (OR 96.2) A two player extensive game.

Solution. The strategic form of the game is (best responses highlighted)

P1 \ P2	L	R
A	0,0	2,1
B	1,2	1,2

The NE are BL and AR. □

Exercise 2.10 Show that (when the game is extended to mixed strategies) every finite extensive game with perfect information has a NE equilibrium.

Solution. For every finite extensive game with perfect information we can define it's strategic form using Definition 4.5 of the section notes (Definition 94.1 in OR). We know that the strategic form has a NE. It is straightforward that this equilibrium is also a NE of the extensive game. □

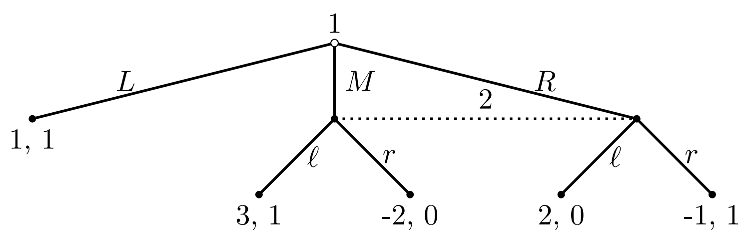
Exercise 2.13 A childs action a (a number) affects both her own private income $c(a)$ and her parents' income $p(a)$; for all values of a we have $c(a) < p(a)$. The child is selfish: she cares only about the amount of money she has. Her loving parents care both about how much money they have and how much their child has. Specifically, model the parents as a single individual whose preferences are represented by a payoff equal to the smaller of the amount of money they have and the amount of money the child has. The parents may transfer money to the child. First the child takes an action, then the parents decide how much money to transfer.

1. Show that in a subgame perfect equilibrium the child takes an action that maximizes the sum of her private income and her parents income.
   Solution. Given a child's action a the parents' problem is
   $$\max_{t \geq 0} \min\{p(a) - t, c(a) + t\}\,.$$
   As $p(a) > c(a)$, the optimal choice $t^\star(a)$ satisfies
   $$p(a) - t^\star(a) = c(a) + t^\star(a) \implies t^\star(a) = \frac{c(a) + p(a)}{2}\,.$$
   Given the parents' behavior, the child's problem is
   $$\max_a c(a) + t^\star(a) = \max_a \frac{c(a) + p(a)}{2}\,.$$
   Hence the child takes an action that maximizes the sum of her private income and her parents income. □
2. Suppose $c(a) = 1 + 2a - a^2$ and $p(a) = 100 + 2a - a^2$ (so both payments are maximized at $a = 1$). Construct a Nash equilibrium of this game in which the child chooses $a = 2$.
   Solution. If $a = 1$ we have $c(a) = 2$ and $p(a) = 101$; if $a = 2$ we have $c(a) = 1$ and $p(a) = 100$. Take the following strategy profile:
   $$\tilde{a} = 2\,, \quad \tilde{t}(a) = \begin{cases} \frac{99}{2} & \text{if } a = 2 \\ 0 & \text{otherwise.} \end{cases}$$
   It is easy to check that the strategy profile is a NE. □

How to find the set of sequential equilibria This is a small description of the steps I take to find the set of sequential equilibria. If is by no means the only form of doing it. I like it because it is explicit and has a well-defined order. The steps are:

1. Define each player's best response for each information set given the other player's strategies *and* the player's belief at that information set.
2. For each region of beliefs in which players' actions are the same, proceed by backwards induction to check that, when everyone is acting according to best responses, beliefs do not contradict Bayes' Rule. When Bayes' Rule cannot be used (because the computation yields a $\frac{0}{0}$) no belief violates it.
3. For each case in the previous step for which everyone plays optimally (i.e., according to their best responses) and beliefs do not contradict Bayes' Rule, build a sequence of assessments that satisfies consistency (make sure that all the strategies in the sequence are completely mixed). If such sequence can be constructed, you found a sequential equilibrium.

Exercise 2.24 Find the set of sequential equilibria of the game in Figure 4.8.



Solution. First denote by γ_a the probability that Player 1 plays action a, and by δ the probability that Player 2 plays ℓ. The expected utilities are

$$u_1(L) = 1,\ u_1(M) = 5\delta - 2,\ u_1(R) = 3\delta - 1;\ \text{and}$$
$$u_2(\ell) = \mu,\ u_2(r) + 1 - \mu\,.$$

Hence Player 1 prefers to play L if $\delta \leq 3/5$ and M if $\delta \geq 3/5$ (R is never an option), and Player 2 prefers to play ℓ if $\mu \leq 1/2$

We proceed by backwards induction and analyze sequential rationality for every possible case of μ. Note that, when $\gamma_M + \gamma_R > 0$, Bayes Rule implies that

$$\mu = \frac{\gamma_M}{\gamma_M + \gamma_R}$$

1. If $\mu > 1/2 \implies$ P2 chooses ℓ ($\delta = 1$) $\implies$ P1 chooses M ($\gamma_M = 1$) $\implies$ $\mu = 1$ satisfies sequential rationality. ✓

2. If $\mu < 1/2 \implies$ P2 chooses r ($\delta = 0$) $\implies$ P1 chooses L ($\gamma_L = 1$) $\implies$ μ cannot be computed using Bayes' Rule. ✓
3. If $\mu = 1/2$, there are two possibilities:
   (a) $\gamma_M = \gamma_R > 0$. This is impossible as R is never a best response for Player 1.
   (b) $\gamma_M = \gamma_R = 0$. In this case $\gamma_L = 1$, which only happens if $\delta \leq 3/5$, which is coherent with $\mu = 1/2$. ✓

Now we need to check consistency in each case. Let β denote the profile of behavioral strategies, and take $\varepsilon^k \to 0$.

1. $\mu = 1$, $\beta = ((0,1,0),(1,0))$: Take the following sequences

$$\beta^k = \left(\left(\varepsilon^k, 1 - 2\varepsilon^k, \varepsilon^k\right), \left(1 - \varepsilon^k, \varepsilon^k\right)\right) \text{ ; and}$$
$$\mu^k = \frac{1 - 2\varepsilon^k}{1 - \varepsilon^k} .$$

   As β^k is completely mixed, $\beta^k \to \beta$, $\mu^k \to \mu$, and μ^k is computed from β^k using Bayes' Rule, then (β, μ) is a sequential equilibrium.
2. $\mu < 1/2$, $\beta = ((1,0,0),(0,1))$. Take the following sequences

$$\beta^k = \left(\left(1 - 2\varepsilon^k - (\varepsilon^k)^2, \mu\varepsilon^k + (\varepsilon^k)^2, (1-\mu)\varepsilon^k\right), \left(1 - \varepsilon^k, \varepsilon^k\right)\right) \text{ ; and}$$
$$\mu^k = \frac{\mu + \varepsilon^k}{1 + \varepsilon^k} .$$

   As β^k is completely mixed, $\beta^k \to \beta$, $\mu^k \to \mu$, and μ^k is computed from β^k using Bayes' Rule, then (β, μ) is a sequential equilibrium. The term $(\varepsilon^k)^2$ ensures that the strategy is totally mixed when $\mu = 0$.

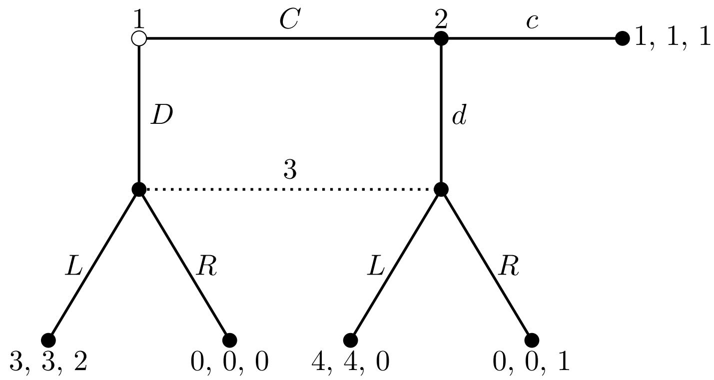
3.(b) $\mu = 1/2$, $\beta = ((1,0,0),(\delta, 1-\delta))$ for $\delta \leq 3/5$. Take the following sequences

$$\beta^k = \left(\left(1 - 2\varepsilon^k, \varepsilon^k, \varepsilon^k\right), \left(\delta + \varepsilon^k, (1-\delta) - \varepsilon^k\right)\right) \text{ ; and}$$
$$\mu^k = \frac{1}{2} .$$

As β^k is completely mixed, $\beta^k \to \beta$, $\mu^k \to \mu$, and μ^k is computed from β^k using Bayes' Rule, then (β, μ) is a sequential equilibrium.

□

Exercise 2.25 Find the set of sequential equilibria of the Selten's Horse



Solution. Let δ_1 be Player 1's probability of playing C, δ_2 Player 2's probability of playing c (conditional on reaching her information set), and δ_3 Player 3's probability of playing L (conditional on reaching her information set). Also, denote by μ Player 3's belief that the history is D conditional on reaching her information set. The player's expected utilities are

$$\begin{aligned} &u_1(D) = 3\delta_3,\ u_1(C) = \delta_2 + 4\delta_3 - 4\delta_2\delta_3 \\ &u_2(c) = 1,\ u_2(d) = 4\delta_3 \\ &u_3(L) = 2\mu,\ u_3(R) = 1 - \mu \end{aligned}$$

hence Player 1 plays D if $3\delta_3 \geq \delta_2 + 4\delta_3 - 4\delta_2\delta_3$, Player 2 plays c if $\delta_3 \leq 1/4$, and Player 3 plays L if $\mu \geq 1/3$. By Bayes' Rule, whenever possible we have

$$\mu = \frac{1-\delta_1}{1-\delta_1+\delta_1(1-\delta_2)}\,.$$

We proceed by backwards induction and analyze sequential rationality for every possible case of μ.

1. If $\mu > 1/3 \implies$ P3 chooses L ($\delta_3 = 1$) $\implies$ P2 chooses d ($\delta_2 = 0$) $\implies$ P1 chooses C ($\delta_1 = 1$) $\implies \mu = 0$, a contradiction.
2. If $\mu < 1/3 \implies$ P3 chooses R ($\delta_3 = 0$) $\implies$ P2 chooses c ($\delta_2 = 1$) $\implies$ P1 chooses C ($\delta_1 = 1$) $\implies \mu$ cannot be computed using Bayes' Rule. ✓
3. If $\mu = 1/3$, there are two possibilities:
   (a) $2(1-\delta_1) = \delta_1(1-\delta_2) > 0$. This implies $\delta_2 < 1$ and $\delta_1 \in (0,1)$. $\delta_2 < 1$ is possible since Player 3 is indifferent between actions. For $\delta_1 \in (0,1)$ we need Player 1 to be indifferent between actions. This happens if

$$3\delta_3 = \delta_2 + 4\delta_3 - 4\delta_2\delta_3 \iff \delta_2 = \frac{\delta_3}{4\delta_3 - 1}\,.$$

For $\delta_2 \in [0,1]$ we need $\delta_3 \geq 1/3$, But in this case Player 2 prefers to play d, hence $\delta_2 = 0$, and $\delta_3/4\delta_3 - 1 > 0$. Therefore this situation is impossible.

(b) $(1-\delta_1) = \delta_1(1-\delta_2) = 0$. This implies $\delta_1 = 1$ and $\delta_2 = 1$. For $\delta_2 = 1$ we need $\delta_3 \leq 1/4$, and as $\delta_2 = 1$ for $\delta_1 = 1$ we need $3\delta_3 \leq 1 + 4\delta_3 - 4\delta_3 \iff \delta_3 \leq 1/3$. Hence, this situation is possible with $\delta_3 \leq 1/4$. ✓

Now we need to check consistency in each case. Let β denote the profile of behavioral strategies, and take $\varepsilon^k \to 0$.

2. $\mu < 1/3$, $\beta = ((1,0),(1,0),(0,1))$. Take the following sequences

$$\beta^k = \left(\left(1-\varepsilon^k, \varepsilon^k\right), \left(1 - \frac{\varepsilon^k(1-\mu)}{(1-\varepsilon^k)\mu + (\varepsilon^k)^2}, \frac{\varepsilon^k(1-\mu)}{(1-\varepsilon^k)\mu + (\varepsilon^k)^2}\right), \left(\varepsilon^k, 1-\varepsilon^k\right)\right) \text{ ; and}$$
$$\mu^k = \frac{(1-\varepsilon^k)\mu + (\varepsilon^k)^2}{1-\varepsilon^k + (\varepsilon^k)^2}.$$

As β^k is completely mixed, $\beta^k \to \beta$, $\mu^k \to \mu$, and μ^k is computed from β^k using Bayes' Rule, then (β, μ) is a sequential equilibrium. The term $(\varepsilon^k)^2$ assures that strategies are well defined when $\mu = 0$.

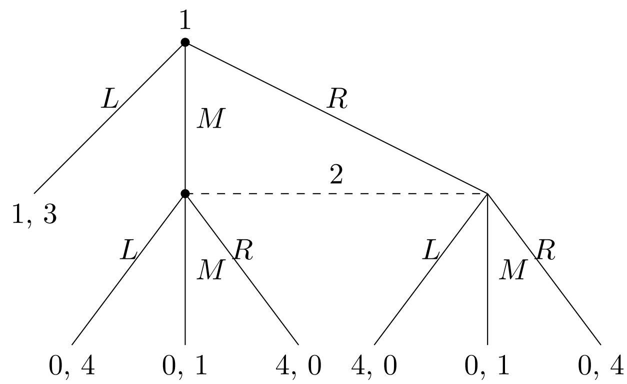
3.(b) $\mu = 1/3$, $\beta = ((1,0),(1,0),(\delta_3, 1-\delta_3))$ with $\delta_3 \leq 1/4$. Take the following sequences

$$\beta^k = \left(\left(1-\varepsilon^k, \varepsilon^k\right), \left(1 - \frac{2\varepsilon^k}{(1-\varepsilon^k)}, \frac{2\varepsilon^k}{(1-\varepsilon^k)}\right), \left(\delta_3 + \varepsilon^k, 1-\delta_3-\varepsilon^k\right)\right) \text{ ; and}$$
$$\mu^k = \frac{1}{3}.$$

As β^k is completely mixed, $\beta^k \to \beta$, $\mu^k \to \mu$, and μ^k is computed from β^k using Bayes' Rule, then (β, μ) is a sequential equilibrium.

□

Exercise 2.26 Find the set of sequential equilibria of the following game.



Solution. Let δ_a denote Player 1's probability of playing action a, and γ_a denote Player 2's probability of playing action a. Also, denote by μ Player 2's belief that the history is M when she has to play. Expected utilities are

$$u_1(L) = 1,\ u_1(M) = 4\gamma_R,\ u_1(R) = 4\gamma_L$$
$$u_2(L) = 4\mu,\ u_2(M) = 1,\ u_2(R) = 4(1-\mu)$$

Player 1 plays L if $\gamma_R \leq 1/4$ and $\gamma_L \leq 1/4$, plays M if $\gamma_R \geq 1/4$ and $\gamma_R \geq \gamma_L$, and plays R if $\gamma_L \geq 1/4$ and $\gamma_L \geq \gamma_R$. Player 2 plays R if $\mu \leq 1/2$ and L if $\mu \geq 1/2$; playing M is never a best response.

We proceed by backwards induction and analyze sequential rationality for every possible case of μ.

- $\mu > 1/2 \implies$ P2 chooses L ($\gamma_L = 1$) $\implies$ P1 plays R ($\delta_R = 1$) $\implies$ $\mu = 0$, a contradiction.
- $\mu < 1/2 \implies$ P2 chooses R ($\gamma_R = 1$) $\implies$ P1 chooses M ($\delta_M = 1$) $\implies$ $\mu = 1$, a contradiciton.
- $\mu = 1/2$. There are two possible cases:
  1. $\delta_M = \delta_R = 0$. As M is never a best response for P2 (when $\mu = 1/2$ P2 is indifferent only between L and R), we must have $\gamma_L + \gamma_R = 1$. Hence we cannot have $\gamma_R \leq 1/4$ and $\gamma_L \leq 1/4$. This implies $\delta_L = 0$, a contradiction.
  2. $\delta_M = \delta_R > 0$. As M is never a best response for P2, $\gamma_M = 0$. For P1 to be indifferent between M and R (so $\delta_M, \delta_R > 0$) we need $\gamma_L = \gamma_R$. This implies $\gamma_L = \gamma_R = 1/2$. In this case P1 is indifferent between M and R but L yields strictly less utility, hence $\delta_M = \delta_R = 1/2$. ✓

As we have only one one candidate for sequential equilibrium and we know that a sequential equilibrium exists (Proposition 4.23 of the section notes), then this candidates has to be a sequential equilibrium. The profile of behavioral strategies is $\beta = ((0, 1/2, 1/2), (1/2, 0, 1/2))$, and the belief is $\mu = 1/2$. To build a sequence, take $\varepsilon^k \to 0$ and define

$$\beta^k = \left(\left(\varepsilon^k, \frac{1}{2} - \varepsilon^k, \frac{1}{2}\right); \left(\frac{1}{2} - \varepsilon^k, \varepsilon^k, \frac{1}{2}\right)\right); \text{ and}$$

$$\mu^k = \frac{1 - \varepsilon^k}{2(1 - \varepsilon^k)}.$$

$\square$

Section 3 : Repeated Games

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These notes are based in the corresponding chapters of Martin Osborne and Ariel Rubinstein's book "A Course in Game Theory", which I refer to as OR. Some ideas were also taken from Martin Osborne's book "An introduction to Game Theory", which I refer to as O.

Summary

In this section we study repeated games. The main insight from these types of games is that a player's present behavior may affect other players' future behavior, which can work as a commitment mechanism. In repeated games, players may choose some actions that are not part of the best response of the strategic game in order to avoid other players' behavior in the future.

3.1 Repeated Games

The idea of studying repeated games is to examine the logic on long-term interaction. While in games that are played only once a players can react to other players' actions only during the game, in repeated games this reaction can happen in future versions of the game.

We will study repeated games by a mixture between general definitions and applications to the repeated *prisoner's dilemma*. The one-time version of this game is shown in Figure 3.1, and has D, D as its unique Nash Equilibrium.

	C	D
C	3,3	0,4
D	4,0	1,1

Figure 3.1: The prisoner's dilemma.

3.1.1 Setup

Recall that an N-player game with perfect monitoring is a tuple $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where N is a set of players, A_i is the set of Player i's actions (with $A = \times_{i\in N} A_i$) and $u_i : A \to \mathbb{R}$ represent Player i's preferences over profiles of actions. A repeated game is a game G that is played repeatedly over time.

To define preferences over games that are played repeatedly, we assume that there is a common discount factor $\delta \in (0, 1)$. Then for any stream of payoffs $(\omega_t)_{t=1}^T$ (where $T \in \mathbb{N} \cup \{\infty\}$), each player's preferences over streams are represented by the discounted sum

$$V = (1-\delta)\sum_{t=1}^{T} \delta^{t-1}\omega_t \, .$$

Given this, we define a repeated game as follows.

Definition 3.1. Given the strategic game G and a discount factor δ

- For $T \in \mathbb{N}$, a **T-period repeated game of G**, denoted $G^\delta(T)$, is an extensive game with perfect information and simultaneous moves $\langle N, H, P, (u_i)_{i\in N}\rangle$ in which
  - $H = \{\emptyset\} \cup \left(\cup_{t=1}^T A^t\right)$;
  - $P(h) = N$ for any $h \in H \setminus Z$; and
  - $u_i = (1-\delta)\sum_{t=1}^T \delta^{t-1} u_i^t$, where u_i^t is the utility of Player i from the game player in period t.
- An **infinitely repeated game of G**, denoted $G^\delta(\infty)$, is an extensive game with perfect information and simultaneous moves $\langle N, H, P, (u_i)_{i\in N}\rangle$ in which
  - $H = \{\emptyset\} \cup \left(\cup_{t\in N} A^t\right) \cup A^\infty$, where A^∞ is the set of infinite sequences of action profiles;
  - $P(h) = N$ for any $h \in H \setminus Z$; and
  - $u_i = (1-\delta)\sum_{t=1}^\infty \delta^{t-1} u_i^t$, where u_i^t is the utility of Player i from the game player in period t.

We define strategies in the usual way

Definition 3.2. Given a game $G^\delta(T)$ ($G^\delta(\infty)$) and $i \in N$, Player i's **strategy** is a behavioral strategy $\sigma_i = (\sigma_i^t)_{t=1}^T$ ($\sigma_i = (\sigma_i^t)_{t\in N}$) where $\sigma_i^1 \in \Delta A_i$ and for any $t > 1$

$$\sigma_i^t : (A^s)_{s=1}^{t-1} \to \Delta A_i,$$

where $A^s \in A$ is the profile of actions taken in period $s < t$.

3.1.2 Strategies as Machines

Although strategies can be very complex in this setting, we can develop a language in which to describe conveniently the structure of the equilibria that we find. Specifically, we can think of strategies as *machines* (or *automaton*) who prescribe decisions according to simple rules.

Definition 3.3. Given $G^\delta(T)$ ($G^\delta(\infty)$) a **machine** has the following components

- a set Q_i (the set of *states*);
- an *initial state* $q_i^0 \in Q_i$;
- a *output function* $f_i : Q_i \to A_i$ that assigns a function to every state; and
- a *transition function* $\tau_i : Q_i \times A \to Q_i$, that assigns a state to every pair consisiting of a state and an action profile.

The idea is the following. There is an unrestricted set of states Q_i and an initial state $q_i^0 \in Q_i$. Then, in the first period the machine chooses $f_i(q_i^0)$. Given the action profile in the first period a^0, the state changes to $q_i^1 = \tau_i(q_i^0, a^0)$. Then machine chooses $f_i(q_i^1)$ in the next period, the state updates again, and so on.

Exercise 3.4. Take the repeated prisoner's dilemma, and define the following strategies:

- *Grim Trigger*: Play C as long as the other player has always played C in the past. If not, play D.
- *Limited punishment (k periods)*: Start playing C and after play C if the other player played C in the previous period. If the other player plays D, then play D for the next k periods. Then start again.
- *Tit-for-tat*: Play C in the first period. Then in every period mimic the action the other played in the previous period.

Define machines describing these strategies.

3.1.3 Folk Theorem for Repeated Games

Machines are useful not only because they allow us to describe strategies but also because simple machines help us prove what are called "Folk Theorems". Folk theorems describe the set of possible equilibrium payoffs in repeated games. Here we focus on one specific Folk Theorem, which we will show applied to the prisoner's dilemma.

For a finitely repeated prisoner's dilemma, the Folk theorem is a negative result: we cannot escape from the unique (and suboptimal) Nash equilibrium. This result hold for any game with a unique Nash equilibrium.

Proposition 3.5. *If the strategic game G has a unique Nash equilibrium payoff profile, then for any $T \in \mathbb{N}$, the action profile chosen after any history in any subgame perfect equilibrium of $G^\delta(T)$ is the Nash equilibrium of G.*

Proof. The outcome in any subgame that starts in period T of any subgame perfect equilibrium of $G^\delta(T)$ is the Nash equilibrium of G. Thus each player's payoff in the last period of the game is independent of history. Consequently in any subgame that starts in period $T-1$ the action profile is a Nash equilibrium of G. An inductive argument completes the proof. □

This result basically tells us that there are not any changes in behavior in finitely repeated games when the static game has a unique Nash equilibrium. The reason is that the uniqueness of the Nash equilibrium makes last period's payoffs independent of the history, which also makes the second-to-last period payoffs independent of the history, and so on.

The Folk theorem for infinitely repeated games tells us a story completely different than the one in finitely repeated games. Before establishing the theorem we need the following auxiliary result.

Lemma 3.6. *A strategy profile is a subgame perfect equilibrium of $G^\delta(\infty)$ if and only if no player can gain by deviating in a single period after any history.*

Proof. Let s be a strategy profile and $(v^t)_{t\in N}$ be the infinite sequence of payoffs induced by s. Also, let

$$U_i(s) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} v_i^t\,;$$

and for any history $h = (a^i, \dots, a^T)$, let

$$W_i(s,h) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} u_i(a^{T+t})\,,$$

where $(a^{T+t})_{t\in\mathbb{N}}$ is induced by s. That is, $W_i(s,h)$ is Player i's payoff following history h given the strategy profile s.

If a player can gain by a one-period deviation then the strategy profile is obviously not a subgame perfect equilibrium.

Now assume that no player can gain by a one-period deviation from s after any history but there is a history h after which Player i can gain by switching to the strategy $s_i^\star$. Without loss of generality assume that $h = \emptyset$. As $\delta < 1$ there is some period T such that the strategy that follows $s_i^\star$ until T and then follows s_i is still a profitable deviation for Player i, call this strategy s_i' and $s' = (s_i', s_{-i})$. As $h = \emptyset$, we have that $U_i(s') > U_i(s)$.

Let $h^t = (a^1, \ldots, a^t)$ be the sequence of outcomes of G induced by s' in the first t periods of the repeated game. Then we have

$$U_i(s') = (1-\delta)\sum_{t=1}^{T} \delta^{t-1} u_i(a^t) + W_i(s, h^T)$$

Now, since no player can gain by deviating in a single period after any history, Player i cannot gain by deviating from s_i in the first period of the subgame that follows the history h^{T-1}. Thus

$$U_i(s') \leq (1-\delta)\sum_{k=1}^{T-1} \delta^{t-1} u_i(a^t) + W_i(s, h^{T-1}) \,.$$

Now take the strategy s'' that follows s' until $T-1$ and after that follows s. Note that

$$U_i(s'') = (1-\delta)\sum_{k=1}^{T-1} \delta^{t-1} u_i(a^t) \,,$$

so $U_i(s'') \geq U_i(s')$. However, as Player i cannot gain by deviating from s_i in the first period of the subgame that follows the history h^{T-2} (i.e., s'' is not a profitable deviation) we have that

$$U_i(s'') \leq (1-\delta)\sum_{k=1}^{T-2} u_i(a^t) + W_i(s, h^{T-2}) \,.$$

Therefore

$$U_i(s) \leq (1-\delta)\sum_{k=1}^{T-2} u_i(a^t) + W_i(s, h^{T-2}) \,.$$

Continuing to work backwards period by period leads to the conclusion that

$$U_i(s') \leq W_i(s, \emptyset) = U_i(s) \,.$$

Contradicting the assumption that s'_i is a profitable deviation for Player i. □

A candidate for a Folk theorem would be one in which any payoff that comes from a profile of actions is available. However, this is clearly not the case. Take for example the prisoner's dilemma, and suppose we want to generate the payoffs coming from the action profile C, D, which yields a payoff of 0 to Player 1. Then Player 1 can increase her payoff by switching to D, even if Player 2 punishes that change by deviating to C. In the worst possible scenario, Player 1 can still achieve a payoff of 1. We formalize this idea introducing the **minmax payoff** of Player i by

$$v_i = \min_{a_{-i}\in A_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i}) \,. \tag{3.1}$$

We say that a payoff profile $w = (w_i)_{i \in N}$ is **enforceable** if $w_i \geq v_i$ for every $i \in N$, and **strictly enforceable** if $w_i > v_i$ for every $i \in N$. Nop we establish the main result of this part of the materials.

Theorem 3.7. *Let (w_1, w_2) be a strictly enforceable payoff profile derived from a profile of actions $a^\star = (a_1^\star, a_2^\star)$. then there exists a $\underline{\delta} \in (0,1)$ such that for all $\delta > \underline{\delta}$ there is a subgame perfect equilibrium of $G^\delta(\infty)$ in which the discounted average payoffs of each Player i is u_i.*

Proof. For each Player i, take Player $i's$ minmax action $b_i \in \arg\min_{a_i \in A_i} \max_{a_{-i} \in A_{-i}} u_{-i}(a_i, a_{-i})$. Now consider the following strategy σ_i for Player i: "Start playing $a_i^\star$, and continue to use it as long as both the profile of actions is $a^\star$ in every period. If not, play b_i."

Let a_i' be the one-time best response to $a_{-i}^\star$, i.e., $a_i' \in \arg\max_{a_i \in A_i} u_i(a_i, a_{-i}^\star)$, and let $w_i' = \max_{a_i \in A_i} u_i(a_i, a_{-i}^\star)$. Now take the one-time deviation for Player i of playing w_i' in period t and σ_i afterwards, and wothout loss of generality set $t = 1$. The payoff from this deviation is

$$(1-\delta) u_i(a_i', a_{-i}^\star) + \delta(1-\delta) \sum_{t=2}^{\infty} \delta_{t-1} v_i = (1-\delta) w_i' + \delta v_i \,,$$

where v_i is defined in (3.1). Then Player i has no incentive to deviate if

$$w_i \geq (1-\delta) w_i' + \delta v_i \iff \delta \geq \frac{w_i' - w_i}{w_i' - v_i} \,.$$

Define

$$\underline{\delta} = \min_{i=1,2} \frac{w_i' - w_i}{w_i' - v_i}$$

Since (w_1, w_2) is strictly enforceable, it is clear that $\underline{\delta} < 1$.

The last part to check that $\sigma = (\sigma_1, \sigma_2)$ is a subgame perfect equilibrium with the desired payoffs is to check that there are no one-time profitable deviations after any history, which is left as an exercise. $\square$

This result states the set of payoffs that can be sustained by a subgame perfect equilibrium. This set is the set of strictly enforceable payoffs that come from a profile of actions. Using mixed strategies (and a public randomization device) we can also achieve convex combinations of these payoffs. Moreover, this result can also be extended to games with more than two players (after some technical assumptions), but these results are outside of the scope of the course.

We finish this section with an exercise.

Exercise 3.8 (O 443.1)**.** *(Delayed modified grim trigger strategies)* Choose a positive integer k and consider the strategy that chooses D in the first k periods of the game, regardless of the history, and then follows the modified trigger strategy, starting in state $\mathcal{C}$, where the modified trigger strategy is defined as follows.

- $Q_i = \{\mathcal{C}, \mathcal{D}\}$;
- $q_i^0 = \mathcal{C}$;
- $f_i(\mathcal{C}) = C$, $f_i(\mathcal{D}) = D$; and
- $\tau_i(\mathcal{C}, (C, C)) = \mathcal{C}$, $\tau_i(\mathcal{X}, (X, Y)) = \mathcal{D}$ for any $(\mathcal{X}, (X, Y)) \neq (\mathcal{C}, (C, C))$.

Find a range of values for the discount factor for which the strategy profile in which each player uses this strategy is a subgame perfect equilibrium.

Section 3 : Repeated Games

Suggested Solutions

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Exercise 3.4 The machines can be defined as follows (for each $a \in A$, the first action represents my action and the second one the action of the other player):

- Grim Trigger:
  - $Q = \{q_1, q_2\}$;
  - initial state is q_1;
  - $f(q_1) = C$, $f(q_2) = D$; and
  - $\tau(q_1, (\cdot, C)) = q_1$, $\tau(q_1, (\cdot, D)) = q_2$, and $\tau(q_2, a) = q_2$ for every $a \in A$
- Limited Punishment:
  - $Q = \{q_0, q_1, \ldots, q_k\}$;
  - initial state is q_0;
  - $f(q_0) = C$, $f(q_i) = D$ for $i = 1, \ldots, k$; and
  - $\tau(q_0, (\cdot, C)) = q_0$, $\tau(q_0, (\cdot, D)) = q_1$, $\tau(q_i, a) = q_{i+1}$ for every $a \in A$ and $i \in \{1, \ldots, k-1\}$, and $\tau(q_k, a) = q_0$ for every $a \in A$.
- Tit-for-tat:
  - $Q = \{q_1, q_2\}$;
  - initial state is q_1;
  - $f(q_1) = C$, $f(q_2) = D$; and
  - $\tau(q, (\cdot, C)) = q_1$ and $\tau(q, (\cdot, D)) = q_2$ for every $q \in Q$.

Exercise 3.8 Any deviation to C at the start of a subgame following a history of length $k - 1$ or less reduces a player's payoff and has no impact on the subsequent outcomes. No deviation in the first period of a later subgame is profitable if and only if $\delta \geq 1/2$, by the argument for the modified grim trigger strategy. Thus the strategy pair is a subgame perfect equilibrium if and only if $\delta \geq 1/2$.

Section 4 : Bargaining

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These notes are based in the corresponding chapters of Martin Osborne and Ariel Rubinstein's book "A Course in Game Theory", which I refer to as OR. Some ideas were also taken from Martin Osborne's book "An introduction to Game Theory", which I refer to as O.

Summary

In this section we study the problem of bargaining, focusing on the case with only two parts. The idea is simple: there is a set of possible outcomes and players' preferences over those outcomes are not aligned. The typical example is when players have to split a dollar. The main question is to select which outcome(s) seem reasonable. We take two different approaches to solve this problem: the axiomatic approach and the strategic approach.

4.0 Motivation

The main problem about the study of bargaining is that a general approach to the problem may be insufficient. We show it in the following exercise.

Exercise 4.1. Suppose two players, which we call 1 and 2, play the following strategy game. Player 1 chooses a number $x \in [0, 1]$ and Player 2 simultaneously chooses a number $y \in [0, 1]$. If $x + y \leq 1$ then Player 1 receives x and Player 2 receives y; if $x + y > 1$ then both players get zero. Show that any combination $(x^\star, y^\star)$ satisfying $x^\star + y^\star = 1$ is a Nash Equilibrium.[1]

[1] It is possible to show that refinements as trembling hand perfection or evolutionary stability do not reduce the set of Nash equilibria.

4.1 The Axiomatic Approach

The axiomatic approach to bargaining dates back to Nash's (1950)[2] seminal paper. This approach starts from a very general bargaining problem. Then introduces some properties (axioms) for the bargaining outcome that seem "reasonable" to study which how to characterize the outcomes that satisfy such properties.

4.1.1 Setup

A bargaining situation is defined as follows.

Definition 4.2. A **bargaining situation** is a tuple $\langle N, A, D, (\succsim_i)_{i\in N}\rangle$ where

- N is a set of players
- A is a set of outcomes/agreements
- D is a disagreement outcome
- $\succsim_i$ is Player i's preference relation on lotteries over $A \cup \{D\}$.

The idea of the problem is that players have to all agree over an outcome $a \in A$, and if someone disagrees about it then the outcome is D.

We simplify our lives by making the following assumptions

Assumption 4.3. There are only two players, i.e., $N = \{1, 2\}$.

Assumption 4.4. $\succsim_i$ has an expected utility representation for every $i \in N$.

Assumption 4.3 is just to simplify the notation and exposition. As in several settings, we can restrict our focus to situations with only two players without loosing insight about the results.

Assumption 4.4 implies that for each player there is a Bernoulli function $u_i : A \cup \{D\} \to \mathbb{R}$. This allow us to simplify the exposition of the bargaining situation and to make it easier to study. In particular, for each possible outcome $a \in A$ we can define a vector $(u_1(a), u_2(a))$, and similarly we can define the vector $(u_1(d), u_2(d))$. By doing this, we move from an abstract set of alternatives A to the set of utilities.

In simple, instead of focusing on outcomes, we think of bargaining as players choosing a pair of payoffs $(s_1, s_2) \in S$, where the set S is given by

$$S = \left\{(s_1, s_2) \in \mathbb{R}^2 : s_1 = u_1(a) \text{ and } s_2 = u_2(a) \text{ for some } a \in A\right\} .$$

[2] Nash Jr, J. F. (1950). The bargaining problem. *Econometrica: Journal of the econometric society*, 155-162.

If players disagree, their payoffs are given by $d = (u_1(D), u_2(D))$. Note that this simplified situation in which we look only at the sets S and d still contains *all the relevant information* about the bargaining situation. In what follows we will study the axiomatic approach to bargaining by looking only at these two sets.

We make some structural assumptions about the pairs of utilities than can be obtained in equilibrium.

Assumption 4.5. S is a compact and convex set.

The assumption of compactness makes the situation easier to study. Recall that a set is compact if and only if it is closed and bounded.

- As the set is closed we have that limits are achievable, so we will never want to get "very" close to a point that is not in S.
- As the set is bounded no player will get infinite utility.

The assumption about convexity is more technical. With this assumption we define a bargaining problem.

Definition 4.6. A **bargaining problem** is a pair $\langle S, d\rangle$, where S is a convex and compact subset of $\mathbb{R}^2$, $d \in S$, and there is $s \in S$ satisfying $s \gg d$.[3]

The set of all bargaining problems is denoted by B. A couple of comments about the previous definition.

- A bargaining problem does not care about what the options are, only about the achievable set of payoffs.
- There is an outcome that both players prefer over the disagreement.

It will be useful to study a particular case of bargaining problems

Definition 4.7. A bargaining problem $\langle S, d\rangle$ is **symmetric** if

1. $d_1 = d_2$; and
2. $(s_1, s_2) \in S \iff (s_2, s_1) \in S$.

4.1.2 Nash's Axioms and Solutions

Our approach here is, instead of modeling specific characteristics of a bargaining problem, to study the solution of the abstract bargaining problem. Specifically, for every problem we look a solution.

[3] For two vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we write $x \gg y$ whenever $x_1 > y_1$ and $x_2 > y_2$.

Definition 4.8. A **bargaining solution** is a function $f : B \to \mathbb{R}^2$ satisfying $f(\langle S, d\rangle) \in S$.

Simply, a bargaining solution takes a bargaining problem $\langle S, d\rangle$ and returns a unique outcome of that problem.

Given the generality of our bargaining problem, we look for properties (axioms) that seem general enough such that they should hold for every problem. Then we ask for the solution to have these properties. Nash's original axioms are:

(INV) **Invariance to equivalent utility representations** For $i = 1, 2$ take $\alpha_i > 0$ and $\beta_i \in \mathbb{R}$, and define the bargaining problem $\langle S', d'\rangle$, where

$$S' = \{(\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2) : (s_1, s_2) \in S\} \text{ ; and}$$
$$d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)\,.$$

Then for $i = 1, 2$ we have $f_i(\langle S', d'\rangle) = \alpha_i f_i(\langle S, d\rangle) + \beta_i$.
The idea of this axiom is that, as utilities are only ordinal representations, monotone transformations of the utility function should not affect the outcome (we focus only on affine transformations to keep convexity).

(SYM) **Symmetry** If $\langle S, d\rangle$ is symmetric, then $f_1(\langle S, d\rangle) = f_2(\langle S, d\rangle)$.
Intuitively if both players are equal we have no reason to favor one over the other.

(IIA) **Independence of Irrelevant Alternatives** If $\langle S, d\rangle$ and $\langle T, d\rangle$ are bargaining problems such that $S \subset T$ and $f(\langle T, d\rangle) \in S$, then $f(\langle S, d\rangle) = f(\langle T, d\rangle)$.
Intuitively, as long as the chosen outcome remains available, removing alternatives that were not chosen should not change which one we choose.

(WPO) **Weak Pareto Efficiency** If $s, t \in S$ and $t \gg s$, then $f(\langle S, d\rangle) \neq s$.
This axiom tells us that players will not choose an outcome if there is another one in which both are be strictly better.

These four axioms are the ones that Nash proposed to construct a solution to the bargaining problem. Before doing that, we show the following result.

Exercise 4.9. Show that INV implies that it is without loss of generality to restrict B to problems $\langle S, d\rangle$ where (1) $d = (0, 0)$, (2) $S \subset \mathbb{R}^2_+$, and (3) $S \cap \mathbb{R}^2_{++} \neq \emptyset$.

(Given this result, in what follows we refer to B as the set of bargaining problems satisfying these properties)

Using the previous four axioms Nash proposed the following solution.

Proposition 4.10. *The unique bargaining solution satisfying INV, SYM, IIA and WPO is*

$$f^N(\langle S,d\rangle) = \arg\max_{(s_1,s_2)\in S} s_1\, s_2\,.$$

*We call f^N the **Nash's solution**.*

We will prove this result in two steps.

Exercise 4.11. Show that f^N is a well defined bargaining solution (this is, that f^N is a function from B to $\mathbb{R}^2$ where $f(\langle S,d\rangle) \in S$).

Exercise 4.12. Show that f^N is the unique bargaining solution satisfying INV, SYM, IIA and WPO.

4.1.3 Alternative Axioms and Solutions

Although the four axioms presented before may seem reasonable, alternatives have been proposed. In particular, the IIA axiom is specifically problematic. This axiom has been object of debate also in other settings, for example in social choice theory.[4]

In particular, two alternatives have been proposed to IIA. They are

(INM) **Individual monotonicity** For $i = 1, 2$ define $\bar{s}_i = \max_{s \in S} s_i$. Then

1. For any $\langle S, d\rangle$ and $\langle T, d\rangle$ with $S \subset T$ and $\bar{s}_i = \bar{t}_i$ for $i = 1, 2$, then for $i = 1, 2$ we have $f_i(\langle S, d\rangle) \leq f_i(\langle T, d\rangle)$.
2. For any $\langle S, d\rangle$ and $\langle T, d\rangle$ with $S \subset T$ and $\bar{s}_i = \bar{t}_i$ for some player i, we have that $f_j(\langle S, d\rangle) \leq f_j(\langle T, d\rangle)$ for $j \neq i$.

In the word of Kalai and Smorodinsky (who proposed this axiom), the idea is that "If, for every utility level that Player i may demand, the maximum feasible utility level that Player j can simultaneously reach is increased, then the utility level assigned to Player j according to the solution should also be increased".

(STM) **Strong Monotonicity** For any $\langle S, d\rangle$ and $\langle T, d\rangle$ with $S \subset T$ we have

$$f_i(\langle S, d\rangle) \leq f_i(\langle T, d\rangle)$$

for $i = 1, 2$.

In simple, if we add alternatives then players should not be worst off.

IIA can be replaced by these axioms, reaching the following results.

Proposition 4.13. *The unique bargaining solution satisfying SYM, WPO, INV and WMO is*

$$f^{KS}(\langle S, d\rangle) = \left\{ s \in S : \frac{s_1}{\bar{s}_1} = \frac{s_2}{\bar{s}_2} \right\} \cap WPO(S)\,.$$

*Where $WPO(S) = \{s \in S : t \gg s \implies t \notin S\}$. We call f^{KS} the **Kalai-Smorodinsky solution**.*

Proposition 4.14. *The unique bargaining solution satisfying SYM, WPO, and STM is*

$$f^{K}(\langle S, d\rangle) = \{s \in S : s_1 = s_2\} \cap WPO(S)\,.$$

*We call f^K the **Kalai solution**.*

[4] See for example Eric Maskin's working paper "Arrow's Theorem, May's Axioms, and the Borda Rule".

Since time is limited we will not go through the proofs of these results (they are available in the lecture slides). An interesting comment about the Kalai-Smorodinsky solution is that only the first part of the INM axiom is necessary for the result. Moreover, the second condition implies the first one, so we only need one of the two. We present both conditions here because while the first one is weaker (and hence the second is not necessary), the second one is the original one proposed by Kalai and Somorodinsky.

We finish this section with some exercises.

Exercise 4.15 (2019 Midterm)**.** Show that in the standard Nash bargaining problem WPO can be replaced with SIR.

(SIR) **Strict Individual Rationality** In any bargaining problem $\langle S, d\rangle$ we have $f(S,d) \gg d$.

That is, the Nash bargaining solution

$$f^N(S,d) = \arg\max_{\substack{s\in S\\ s\geq d}} (s_1 - d_1)(s_2 - d_2)$$

is the only solution satisfying SYM, SIR, INV and IIA (it is sufficient to show that these four axioms are equivalent to the standard axioms SYM, WPO, INV and IIA).

Exercise 4.16 (2016 Midterm)**.** Let **B** be the set of all convex, compact and comprehensive sets in $\mathbb{R}^2_+$ with nonempty intersection with $\mathbb{R}^2_{++}$.

1. Show that the Kalai bargaining solution $f^K(\langle S, d\rangle)$ does not satisfy INV.

2. Show that the Kalai-Smorodinsky bargaining solution f^{FS} does not satisfy IIA.

4.2 The Strategic Approach

4.2.1 Setup

In the strategic approach we model bargaining as a game of alternating offers between players. For simplicity we assume there are two players, and we call the set of agreements X, which we assume is given by

$$X = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 = 1\}.$$

The game has essentially two stages, that we call (I) and (II), which repeat over time. It starts at $t = 0$ in stage (I), and proceeds as follows.

(I) Player 1 makes an offer $x \in X$. Player 2 can either Accept or Reject x. If Player 2 chooses A then the game ends, and the agreement is x. If Player 2 rejects we move one period forward and a subgame starts at (II).

(II) Player 2 makes an offer $y \in X$. Player 1 can either Accept or Reject y. If Player 1 chooses A then the game ends, and the agreement is y. If Player 1 rejects we move one period forward and a subgame starts at (I).

If no player never accepts, then the outcome is D. We denote by T the set of possible times; this is, T is the set of nonnegative integers. We assume that time is relevant for players; this is, players have preferences over pairs $(x, t) \in X \times T$. We also make several assumptions about the players' preferences.

Assumption 4.17. For every Player $i = 1, 2$, alternatives $x, y \in X$, and times $t, s \in T$ we have

(A1) $(x, t) \succsim_i D$;

(A2) $(x, t) \succsim_i (y, t) \iff x \geq y$;

(A3) $(x, t) \succsim_i (x, t + 1)$, with strict preference if $(x, 0) \succ_i D$;

(A4) If there are sequences $x^n, y^n \in X$ such that $x^n \to x$, $y^n \to y$, and $(x_n, t) \succsim_i (y_n, s)$ for all n, then $(x, t) \succsim_i (y, s)$; and

(A5) $(x, t) \succsim_i (y, t + 1) \iff (x, 0) \succsim_i (y, 1)$.

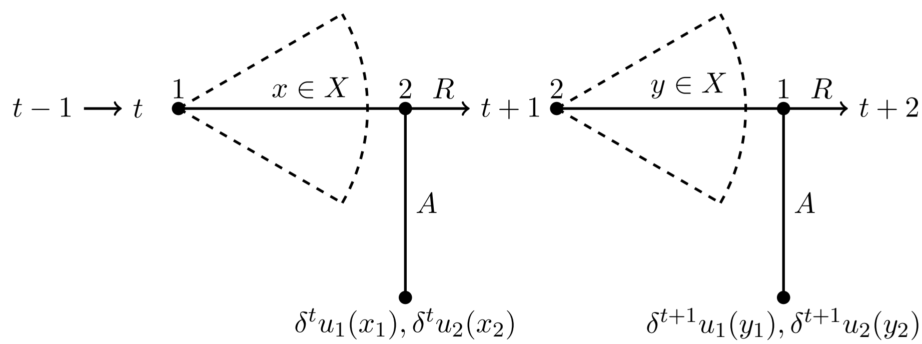
Our assumptions have several implications regarding the properties and representation of the players' preferences. Since this is a course on gme theory, not choice theory, we will only state these results.

Lemma 4.18. *Under assumptions A2 to A4, for $i = 1, 2$ and any $(x, t) \in X \times T$ there is at most one $y \in X$ satisfying $(y, 0) \sim_i (x, t)$. If no such y exists, then $(z, 0) \succ_i (x, t)$ for all $z \in X$.*

Lemma 4.19. *Under assumptions A2 to A5, for $i = 1, 2$ and every $\delta \in (0, 1)$ there exists a continuous and increasing function $u_i : X \to \mathbb{R}$ such that $U_i(x, t) = \delta^t u_i(x_i)$ represents $\succsim_i$.*

In particular, Lemma 4.19 implies that, if we allow for $u_1 \neq u_2$, to set the same discount factor for both players is without loss of generality.

Given that for each Player i the utility representation can be thought as a function of only x_i and T (even before Lemma 4.19), we will write down Player i's utility as a function of only x_i and t. Graphically the game is (with t even)



We add an additional assumption that allows us to uniquely pin down the equilibrium of the game.

Assumption 4.20. For every Player $i = 1, 2$ and alternative $x \in X$ define the cost of delay as $x - u_i^{-1}(\delta u_1(x))$,

(A6) the cost of delay is an increasing function of x_i.

Lemma 4.21. *Under assumptions A2 to A6 there is a unique pair of agreements $x^\star, y^\star$ such that*

$$(x^\star, t) \sim_2 (y^\star, t+1) \text{ and } (y^\star, t) \sim_1 (x^\star, t+1) .$$

In other words, this pair satisfies $u_2(x_2^\star) = \delta u_2(y_2^\star)$ and $u_1(y_1^\star) = \delta u_1(x_1^\star)$.

4.2.2 Subgame Perfect Equilibrium

Our assumptions imply that the game has a unique SPE

Proposition 4.22 (OR 122.1)**.** *Let $x^\star, y^\star$ be the agreements defined in Assumption A6. Under assumptions A1 to A6 the bargaining game with alternating offers has a unique SPE. The strategies in the SPE are stationary and given by*

- *Player 1 always proposes the agreement $x^\star$ and accepts y if and only if $y_1 \geq y_1^\star$.*
- *Player 2 always proposes the agreement $y^\star$ and accepts x if and only if $x_2 \geq x_2^\star$.*

We will only prove that the strategy profile is a SPE, but not that it is unique (the proof is included in the notes for the interested reader).

Exercise 4.23. Show that the strategy profile described in Proposition 4.22 is a SPE.

Proof of Uniqueness in Proposition 4.22. To see why the SPE is unique, first define, for $i = 1, 2$, the game G_i as the subgame starting at the node at which Player i has to make an offer. Also define the payoffs

$$M_i = \sup\{\delta^t u_i(x) : (x,t) \text{ is the outcome of a SPE of } G_i\}; \text{ and}$$
$$m_i = \inf\{\delta^t u_i(x) : (x,t) \text{ is the outcome of a SPE of } G_i\}.$$

Without loss of generality we normalize $u_1(0) = u_2(0) = 0$. We prove the result in three steps:

1. $M_1 = m_1 = u_2(x_1^\star)$, and $M_2 = m_2 = u_2(y_2^\star)$: We make the argument for Player 1; the argument for Player 2 is analogous. Now suppose we are at game G_1 and Player 1 offers x.
   - It is clear that Player 2 will accept x if $u_2(x_2) > \delta M_2$. Hence Player 1 can assure she will obtain a share of the pie equal to $1 - u_2^{-1}(\delta M_2)$.

$$\begin{aligned} m_1 &\geq u_1\left(1 - u_2^{-1}(\delta M_2)\right) \\ m_2 &\geq u_2\left(1 - u_1^{-1}(\delta M_1)\right) \end{aligned} \tag{4.1}$$

   - Also, Player 2 will reject any offer x if $u_2(x_2) < \delta m_2$. Note that if Player 2 gets a utility δm_2, Player 1 gets a utility no greater than $u_1\left(1 - u_2^{-1}(\delta m_2)\right)$,[5]

$$\begin{aligned} M_1 &\leq u_1\left(1 - u_2^{-1}(\delta m_2)\right) \\ M_2 &\leq u_2\left(1 - u_1^{-1}(\delta m_1)\right) \end{aligned} \tag{4.2}$$

   - We show that $M_1 = u(x_1^\star)$. As we have shown that the strategy profile in Proposition 4.22 is a SPE (Exercise 4.23), we have that $M_1 \geq u_1(x^\star)$. Now towards a contradiction suppose $M_1 > u_1(x_1^\star)$, i.e., $x_1^\star < u_1^{-1}(M_1)$. We have that

$$\begin{aligned} x_1^\star &< u_1^{-1}(M_1) \\ &\leq u_1^{-1}\left(u_1\left(1 - u_2^{-1}(\delta m_2)\right)\right) && \text{(by (4.2) for } M_1\text{)} \\ &= 1 - u_2^{-1}(\delta m_2) && \text{(since } u_1^{-1}(u_1(x)) = x\text{)} \\ &\leq 1 - u_2^{-1}\left(\delta u_2\left(1 - u_1^{-1}(\delta M_1)\right)\right) && \text{(by (4.1) for } m_2\text{)} \end{aligned}$$

   Now define the function $f : [0,1] \to \mathbb{R}$ by $f(x_1) = 1 - u_2^{-1}\left(\delta u_2\left(1 - u_1^{-1}(\delta u_1(x_1))\right)\right)$.[6] This function is continuous, decreasing, and satisfies $f(1) < 1$ and $f(x_1^\star) > x_1^\star$ (since $u_1(x_1^\star) < M_1$). Hence there is a value $\hat{x}_1$ such that $\hat{x}_1 = f(\hat{x}_1)$. De-

[5] To see this note that utilities are rival in the share of the pie; this is, to increase one Player's share, and her utility, decreases the other player's utility (for a given period).

[6] Although this function looks like it comes out of nowhere, it is based on the right hand side of the previous inequality, by noticing that M_1 is a utility of Player 1.

fine $\hat{x} = (\hat{x}_1, 1 - \hat{x}_1)$, and $\hat{y}$ satisfying $(\hat{x}, 0) \sim_2 (\hat{y}, 1)$. This implies that $u_2\,(1 - \hat{x}_1) = \delta u_2(1 - \hat{y}_1)$. Replacing the value of $\hat{x}_1$ and after some (simple) algebra we conclude that $\hat{y}_1 = u_1^{-1}(\delta u_1(\hat{x}^1))$. This implies that $u_1(\hat{y}_1) = \delta u_1(\hat{x}_1)$, so $(\hat{y}, 0) \sim_1 (\hat{x}, 1)$. Hence we have that $\hat{x} \neq x^\star$ (as $\hat{x}_1 > x^\star$), $(\hat{x}, 0) \sim_2 (\hat{y}, 1)$, and $(\hat{y}, 0) \sim_1 (\hat{x}, 1)$. This contradicts Lemma 4.21. We conclude that $M_1 = u_1(x^\star)$.

- That $m_1 = u_1(x_1^\star)$ and $M_2 = m_2 = u_2(y_2^\star)$ can be proved analogously.

2. In every SPE of G_1 Player 1's initial proposal is $x^\star$ which is accepted by Player 2: First suppose Player 2 rejects the initial offer. Then we move to G_2, in which Player 2's payoff is $\delta u_2(y_2^\star)$ and Player 1's payoff is $\delta u_1(y_1^\star) < u_1(y_1^\star) < u_1(x_1^\star)$ (the last inequality because $u_1(y_1^\star) = \delta u_1(x_1^\star)$). Hence in any SPE Player 2 accepts the first offer. As $M_1 = u_1(x_1^\star)$ the offer has to be $x^\star$. Similarly we can show that in every SPE of G_2 Player 2's initial proposal is $y^\star$ which is accepted by Player 1.
3. In any SPE of G_1 Player 2 accepts an offer x if and only if $x_2 \geq x_2^\star$: Player 2's rejection leads to G_2, whose present value is $\delta u_2(y_2^\star)$. By the previous step we know that Player 2 accepts $x^\star$. By the one deviation property is clear that she accepts any offer with $x_2 > x_2^\star$ and rejects any offer with $x_2 < x_2^\star$. Similarly we can show that in every SPE of G_1 Player 1 accepts an offer y if and only if $y_1 \geq y_1^\star$.

□

Properties of the SPE The SPE of the game presents the following properties:

- *Stationarity*: The SPE strategies do not depend on the history: at the beginning of any subgame at which a player has to make an offer she makes the same offer, and at the beginning of any subgame at which a player has to accept or reject an offer she follows the same rule.
- *First Mover Advantage*: Player 1 has an advantage for moving first. Intuitively Player 2 will accept any initial offer as long at its value is higher than the value of moving forward. Player 1 uses this to offer the smallest amount that Player 2 will accept.
- *Patience is good*: Given two preferences $\succsim, \succsim'$ over $X \times T$, we say that $\succsim$ is *less patient* than $\succsim'$ is whenever $(y, 0) \sim' (x, 1)$ we have that $(y, 0) \succsim (x, 1)$. When a player becomes less patient and has not the advantage of moving first her share of the pie decreases in equilibrium.

We finish the study of the strategi approach to bargaining with an exercise.

Exercise 4.24. Two pirates are bargaining about how to split a treasure worth 1. They both have the same discount factor $\delta \in (0, 1)$

1. Consider the following situation
   - In the first round Pirate 1 has to offer Pirate 2 a share of the treasure x.
   - If Pirate 2 accepts Pirate 1's offer, she gets x and Pirate 1 gets $1 - x$.
   - Instead of accepting, Pirate 2 can make a counteroffer y to Pirate 1.
   - If Pirate 1 accepts the counteroffer she gets y and Pirate 2 gets $1 - y$ *delayed by one period.*
   - If Pirate 1 rejects Pirate 2's counteroffer they both get zero.

   Model this game as an extensive game using a tree and find the SPE of the game.

2. Now assume that after Pirate 2's offer, Pirate 1 can make another counteroffer. If Pirate 2 accepts such counteroffer the respective payments are *delayed by an extra period*; if not, they both get zero. Model this game as an extensive game using a tree and find the SPE of the game.

3. According with your previous results, what can you say with respect to the effects on bargaining power of starting and finishing the game?

Section 4 : Bargaining

Suggested Solutions

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Exercise 4.9 how that INV implies that it is without loss of generality to restrict B to problems $\langle S, d\rangle$ where (1) $d = (0, 0)$, (2) $S \subset \mathbb{R}^2_+$, and (3) $S \cap \mathbb{R}^2_{++} \neq \emptyset$.

Solution. Take any bargaining problem $\langle S, d\rangle$. By using $\alpha_1 = \alpha_2 = 1$, $\beta_1 = d_1$, and $\beta_2 = d_2$ in the definition of INV we have a new problem $\langle S', d'\rangle$ where $d' = (0, 0)$. Finally, as there is $s' \in S'$ satisfying $s' \gg (0, 0)$ it is clear that $S' \cap \mathbb{R}_{++} \neq \emptyset$. Finally, it is clear that any point $s \in S \setminus \mathbb{R}_+$ is irrelevant as at least one player prefers the disagreement to any of these points. □

Exercise 4.11 Show that f^N is a well defined bargaining solution (this is, that f^N is a function from B to $\mathbb{R}^2$ where $f(\langle S, d\rangle) \in S$).

Solution. Since S is compact and s_1s_2 is continuous a maximizer exists. Moreover, as S is convex and s_1s_2 is strictly quasiconcave this maximizer is unique. □

Exercise 4.12 Show that f^N is the unique bargaining solution satisfying INV, SYM, IIA and WPO.

Solution. Suppose there is another solution $f \neq f^N$ satisfying these axioms, call if f. Let

$$S' = \left\{ \left(\frac{s_1}{f_1^N(\langle S, d\rangle)}, \frac{s_2}{f_2^N(\langle S, d\rangle)} \right) : (s_1, s_2) \in S \right\} .$$

We have that $s'_1 s'_2 \leq 1$ for all $s' \in S'$, so $f^N(\langle S', d\rangle) = 1, 1$.

Since S is convex S' also is, and hence is contained in the set $T = \{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1\}$.

- By WPO and SYM we have that $f(\langle T, d\rangle) = (1, 1)$;
- as $(1, 1) \in S'$ and $S \subset T$ by IIA we have $f(\langle S', d\rangle) = (1, 1)$;
- Finally, by INV for $i = 1, 2$ we have $f_i(\langle S, d\rangle) = f_i^N(\langle S, d\rangle) f_i(\langle S', d\rangle) = f_i^N(\langle S, d\rangle)$.

Hence $f = f^N$, a contradiction. □

Exercise 4.15 Show that in the standard Nash bargaining problem WPO can be replaced with SIR.

(SIR) **Strict Individual Rationality** In any bargaining problem $\langle S, d\rangle$ we have $f(S,d) \gg d$.

That is, the Nash bargaining solution

$$f^N(S,d) = \underset{\substack{s\in S\\ s\geq d}}{\arg\max}(s_1 - d_1)(s_2 - d_2)$$

is the only solution satisfying SYM, SIR, INV and IIA (it is sufficient to show that these four axioms are equivalent to the standard axioms SYM, WPO, INV and IIA).

Solution. Wlog we set $d = 0$, and write $f(S,d) = f(S)$. Also, denote $AX = SYM + INV + IIA$. We prove both directions:

- ($AX + WPO \implies AX + SIR$): Suppose f satisfies SYM, WPO, INV and IIA, then we know that $f = f^N$. By definition there is $s' \in S$ such that $s' \gg d$. Since $f_1^N(S)f_2^N(S) \geq s_1's_2' > 0$, both $f_1^N(S)$ and $f_2^N(S)$ have to be strictly positive, so SIR is satisfied.
- ($AX + SIR \implies AX + WPO$): Suppose f satisfies SYM, SIR, INV and IIA, and let $z = f(S)$. By SIR we have $z \gg 0$. Towards a contradiction, suppose z is not weakly Pareto Optimal. Then exists $s' \in S$ such that $s_1' \geq z_1$, $s_2' \geq z_2$, and $s' \neq z$. let $a_i = {}^{z_i}/_{s_i'}$. Then $a_i \leq 1$ and $a = (a_1, a_2) \neq (1,1)$. Define $T = \{(a_1s_1, a_2s_2) : (s_1, s_2) \in S\}$. We have that (1) $T \subset S$, and (2) $z \in T$. Then by IIA we have that $f(T) = z$. But as $a \neq (1,1)$, by INV we have that $f(S) = ({}^{z_1}/_{a_1}, {}^{z_2}/_{a_2}) \neq z$, a contradiction. Therefore WPO is satisfied.

□

Exercise 4.16 Let **B** be the set of all convex, compact and comprehensive sets in $\mathbb{R}^2_+$ with nonempty intersection with $\mathbb{R}^2_{++}$.

1. Show that the Kalai bargaining solution does not satisfy INV.
2. Show that the Kalai-Smorodinsky bargaining solution does not satisfy IIA.

Solution. Let $co(A)$ refer to the convex hull of A.

1. Take $S = co(\{(0,0),(1,0),(0,1)\})$. Then $f^K(S) = ({}^1/_2, {}^1/_2)$. Now take $S' = \{(2s_1, s_2) : (s_1, s_2) \in S\}$. Then $f^K(S') = ({}^2/_3, {}^2/_3) \neq (1, {}^1/_2)$.
2. Take $S = co(\{(0,0),(1,0),(0,1)\})$. Then $\bar{s}_1 = \bar{s}_2 = 1$ and $f^{KS}(S) = ({}^1/_2, {}^1/_2)$. Now take $S' = co(\{(0,0),({}^1/_2,0),({}^1/_2,{}^1/_2),(0,1)\})$. Then $\bar{s}_1' = {}^1/_2$, $\bar{s}_2' = 1$, and $f^{KS}(S') = ({}^1/_3, {}^2/_3)$. We have that $f^{KS}(S) \in S'$, $S' \subset S$, and $f^{KS}(S) \neq f^{KS}(S')$.

□

Exercise 4.23 Show that the strategy profile described in Proposition 5.21 is a SPE.

Solution. By the one deviation property the strategy profile is a SPE if there are no one-shot profitable deviations. We analyze the four subgames:

1. Player 1 has to make an offer: If she proposes x with $x_1 > x_1^\star$ then the offer is rejected. In the next period Player 2 offers $y^\star$ and Player 1 accepts it, getting $\delta u_1(y_1^\star) < \delta\delta u_1(x_1^\star) < u_1(x_1^\star)$. If she proposes x with $x_1 < x_1^\star$ Player 2 accepts it and Player 1's utility is $u_1(x_1) < u_1(x_1^\star)$. In this subgame there is no profitable deviation.
2. Player 2 has to make an offer: Analogous to the situation of Player 1.
3. Player 1 has to accept or reject an offer y. If she accepts it she gets $u_1(y_1)$; if she rejects it in the next period she proposes $x^\star$ and Player 2 accepts it, so Player 1's utility is $\delta u_1(x_1^\star)$. It is optimal for Player 1 accept any offer y if $u_1(y_1) \geq \delta u_1(x_1^\star)$. As u_1 is increasing this is equivalent to accept an offer y if and only if $y_1 \geq y_1^\star$.
4. Player 2 has to accept or reject an offer x. Analogous to the situation of Player 1.

□

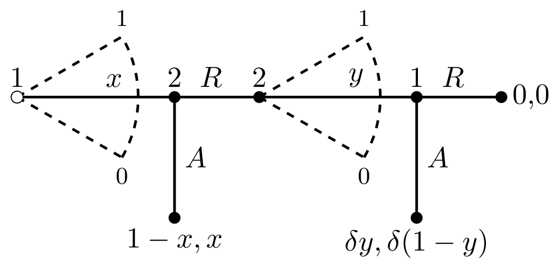
Exercise 4.24 Two pirates are bargaining about how to split a treasure worth 1. They both have the same discount factor $\delta \in (0, 1)$

1. Consider the following situation
   - In the first round Pirate 1 has to offer Pirate 2 a share of the treasure x.
   - If Pirate 2 accepts Pirate 1's offer, she gets x and Pirate 1 gets $1 - x$.
   - Instead of accepting, Pirate 2 can make a counteroffer y to Pirate 1.
   - If Pirate 1 accepts the counteroffer she gets y and Pirate 2 gets $1 - y$ *delayed by one period.*
   - If Pirate 1 rejects Pirate 2's counteroffer they both get zero.

   Model this game as an extensive game using a tree and find the SPE of the game.
2. Now assume that after Pirate 2's offer, Pirate 1 can make another counteroffer. If Pirate 2 accepts such counteroffer the respective payments are *delayed by an extra period*; if not, they both get zero. Model this game as an extensive game using a tree and find the SPE of the game.
3. According with your previous results, what can you say with respect to the effects on bargaining power of starting and finishing the game?

Solution.

1. The tree is the following:

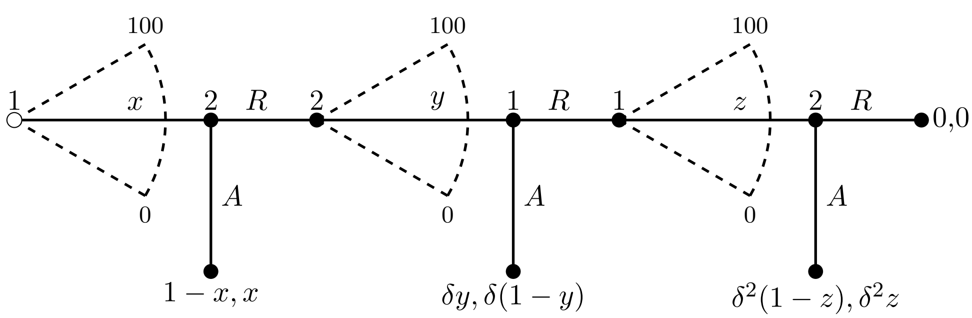


We find the SPE by backwards induction:

- In the last subgame Pirate 1 accepts any offer $y \geq 0$. The payoffs are $\delta y, \delta(1-y)$
- Knowing this, Pirate 2 offers $y^\star = 0$. The payoffs of the subgame starting at Pirate 2's offer are $0, \delta$.
- Given an offer x from Pirate 1, Pirate 2 accepts it if and only if $x \geq \delta$.
- Knowing that Pirate 2 rejects an offer greater than δ, and that the rejection leads to a subgame in which Pirate 1 gets a payoff of zero, Pirate 1 offers $x^\star = \delta$ at the beginning of the game. the payoffs are $1-\delta, \delta$.

In the unique SPE P1 offers $x^\star = \delta$ and accepts any offer y, and P2 accepts if and only if $x \geq \delta$ and offers $y^\star = 0$. The payoffs are $1-\delta, \delta$.

2. The tree is the following:



We find the SPE by backwards induction:

- First, note that the subgame after P1 rejects offer y is exactly the game in part 1. We know that the unique SPE of that game is $z^\star = 0$, which is accepted by P2, and payoffs are $100\delta^2, 0$.
- Hence P1 accepts an offer y only if $\delta y \geq 100\delta^2 \iff y \geq 100\delta$.

- As P2's payoff is decreasing in y (if y is accepted) she offers $y^\star = 100\delta$. The payoffs after this offer are $100\delta^2, 100\delta(1-\delta)$.
- Hence Player 2 accepts an offer x only if $x \geq 100\delta(1-\delta)$.
- As P1's payoff is decreasing in x (if x is accepted) she offers $x^\star = 100\delta(1-\delta)$. The payoffs after this offer are $100(1-\delta(1-\delta)), 100\delta(1-\delta)$. Note that Player 1 has a higher payoff by making this offer than by moving to the next subgame and therefore there is no profitable deviation.

In the unique SPE P1 offers $x^\star = 100\delta(1-\delta)$, accepts y if and only if $y \geq 100\delta$, and offers $z^\star = 0$, while P2 accept an offer x if and only if $x \geq 100\delta(1-\delta)$, offers $y^\star = 100\delta$, and accepts an offer z if and only if $z \geq 0$. The payoffs are $100(1-\delta(1-\delta)), 100\delta(1-\delta)$.

3. Which advantage is more important (either starting or finishing the game) depends on the discount factor δ. If δ is high (i.e., the future is important) then finishing is a higher advantage, but if δ is close to zero (i.e., the future is note very relevant) then starting is a higher advantage. For example, in part 1. Pirate 1 has the "starting advantage" and Pirate 2 has the "finishing advantage". If $\delta > 1/2$ then Pirate 2's payoff is higher, but if $\delta < 1/2$ the Pirate 1 gets the higher share of the treasure.

   Intuitively, the idea is that when the future is not important (δ is close to zero) the Player making the offer can make a low offer because the cost for the other party of rejecting (and delaying the payment) is high. On the other hand, if δ is high the party that makes the last offer can threat to reject offers and get to the last period where she holds all the bargaining power.

$\square$

Section 5 : Review Session

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These notes are based in the corresponding chapters of Martin Osborne and Ariel Rubinstein's book "A Course in Game Theory", which I refer to as OR. Some ideas were also taken from Martin Osborne's book "An introduction to Game Theory", which I refer to as O.

In this section we review exercises from different topics.

5.1 Strategic Games

Exercise 5.1. Suppose α is a NE of the game G, and for Player i there are two different actions $a_i', a_i'' \in A_i$ such that $\alpha_i(a_i') > 0$ and $\alpha_i(a_i'') > 0$. Show that $U_i(\delta_{a_i'}, \alpha_{-i}) = U_i(\delta_{a_i''}, \alpha_{-i})$, where $\delta_{a_i'}$ and $\delta_{a_i''}$ are the probability measures assigning probability one to a_i' and a_i'', respectively. (This is, show that for a player to randomize between two actions in equilibrium she has to be indifferent between the two.)

Solution. To simplify notation, for each action $a_i \in A_i$ write $U_i(a_i) = U_i(\delta_{a_i}, \alpha_{-i})$. We have

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) \sum_{a_{-i} \in A_{-i}} \left(\prod_{\substack{j \in N \\ j \neq i}} \alpha_j(a_j) \right) u_i(a_i, a_{-i}) = \sum_{a_i \in A_i} \alpha_i(a_i) U_i(a_i)\,.$$

Towards a contradiction (and wlog) suppose $U_i(a_i') > U_i(a_i'')$. Define the strategy α_i' for Player i as follows

$$\alpha_i'(a_i) = \begin{cases} \alpha_i(a_i') + \alpha(a_i'') & \text{if } a_i = a_i' \\ 0 & \text{if } a_i = a_i'' \\ \alpha_i(a_i) & \text{otherwise.} \end{cases}$$

Basically, in strategy α_i' Player i never plays a_i'', and instead whenever α_i prescribes to play a_i'', she plays a_i'. We have

$$U_i(\alpha_i', \alpha_{-i}) = \sum_{a_i \in A_i} \alpha_i'(a_i) U_i(a_i) = \sum_{\substack{a_i \in A_i \\ a_i \neq a_i', a_i''}} \alpha_i(a_i) U_i(a_i) + (\alpha_i(a_i') + \alpha_i(a_i''))U_i(a_i')\,.$$

Hence $U_i(\alpha) - U_i(\alpha') = \alpha_i(a_i'')(U_i(a_i'') - U_i(a_i')) < 0$ and α_i' is a profitable deviation for Player i. Therefore α is not a NE. $\square$

Exercise 5.2 (OR 56.4). *(Cournot Duopoly)* Consider the strategic game $\langle\{1,2\},(A_i),(u_i)\rangle$ in which $A_i=[0,1]$ and $u_i(a)=a_i(1-a_1-a_2)$ for $i=1,2$. Show that each player's only rationalizable action is his unique NE action.

Solution. First, note that Player i's best response is $a_i^{BR}(a_{-i})={}^{(1-a_{-i})}\!/\!_2$. Thus the NE is $a^\star=({}^1\!/\!_3,{}^1\!/\!_3)$.

Let Z_i be the set of rationalizable actions, with $m=\inf Z_i$ and $M=\sup Z_i$. As the game is completely symmetric, we have that $Z_1=Z_2=Z$. Take a belief μ of Player i about Player $-i$'s actions, which has expected value $a_\mu\equiv\mathbb{E}_\mu[a_{-i}]$. Player i's best response given this belief is $a_i^{BR}(\mu)={}^{(1-a_\mu)}\!/\!_2$.

As $\operatorname{supp}\mu\subseteq[m,M]$, we have that $a_\mu\in[m,M]$, and therefore $a_i^{BR}(\mu)\in[{}^{(1-M)}\!/\!_2,{}^{(1-m)}\!/\!_2]$. By definition, if $a_i\in Z$ then it is a best response to some belief μ, so $m\geq{}^{(1-M)}\!/\!_2$ and $M\leq{}^{(1-m)}\!/\!_2$. These two inequalities and the fact that $m\leq M$ imply that $m=M={}^1\!/\!_3$. □

Exercise 5.3 (OR 56.3). Find the set of rationalizable actions of each player in the two-player game in Figure 5.1.

	b_1	b_2	b_3	b_4
a_1	0,7	2,5	7,0	0,1
a_2	5,2	3,3	5,2	0,1
a_3	7,0	2,5	0,7	0,1
a_4	0,0	0,-2	0,0	10,-1

Figure 5.1: The game in Exercise 5.3.

Solution. The actions of Player 1 that are rationalizable are a_1, a_2, and a_3; those of Player 2 are b_1, b_2, and b_3. The actions a_2 and b_2 are rationalizable since (a_2,b_2) is a NE. Since a_1 is a best response to b_3, b_3 is a best response to a_3, a_3 is a best response to b_1, and b_1 is a best response to a_1 the actions a_1, a_3, b_1, and b_3 are rationalizable. The action b_4 is not rationalizable since if the probability that Player 2's belief assigns to a_4 exceeds ${}^1\!/\!_2$ then b_3 yields a payoff higher than does b_4, while if this probability is at most ${}^1\!/\!_2$ then b_2 yields a payoff higher than does b_4. The action a_4 is not rationalizable since without b_4 in the support of Player 1's belief a_4 is dominated by a_2.

That b_4 is not rationalizable also follows from it being strictly dominated by the mixed strategy that assigns the probability ${}^1\!/\!_3$ to b_1, b_2, and b_3. □

Exercise 5.4 (2016 Midterm)**.** Consider the Matching Pennies game in which Player 1 has an outside option $x \in (0,1)$ shown in Figure 5.2. Find the set of mixed strategies for

	H	T
$O(ut)$	$x, 0$	$x, 0$
H	1,-1	-1,1
T	-1,1	1,-1

Figure 5.2: The game in Exercise 5.4.

Player 1 that survive iterated elimination of strictly dominated actions. Are these strategies rationalizable? Find the sets of all NE and Trembling Hand Perfect equilibria.

Solution. Let Player 1's strategy be $(p_1, p_2, 1-p_1-p_2)$, and Player 2's strategy be $(q, 1-q)$. Then

$$u_1 = p_1x+p_2(q-(1-q))+(1-p_1-p_2)(-q+(1-q)) = p_1x+p_2(2q-1)+(1-p_1-p_2)(1-2q)\,.$$

If $2q = 1$ then it is optimal for Player 1 to set $p_1 = 1$ ($p_2 = 0$, $1-p_1-p_2 = 0$); if $2q > 1$ then it is optimal for Player 1 to set $1 - p_1 - p_2 = 0$, and if $2q < 1$ then it is optimal for Player 1 to set $p_2 = 0$. Therefore a strategy in which Player 1 plays both H and T with strictly positive probability is strictly dominated. As Player 2's strategy now depends if Player 1 plays H or T with positive probability, we cannot eliminate any Player 2's strategy (we cannot eliminate mixing since is possible when $p_1 = 1$). Furthermore, we cannot eliminate any other strategy. The set of rationalizable strategies is the same as the set of strategies that survive iterated elimination of strictly dominated actions.

There is no pure strategy NE. To allow Player 2 to mix we need

$$U_2(H) = U_2(T) \iff -p_2 + (1 - p_1 - p_2) = p_2 - (1 - p_1 - p_2) \iff 1 - p_1 - p_2 = p_2\,.$$

Since from the first part we know that in any NE either $p_2 = 0$ or $1 - p_1 - p_2 = 0$, from the previous condition we get that in any NE Player 1's strategy is $(1, 0, 0)$. For this strategy to be optimal we need

$$U_1(O) \geq U_1(H) \iff x \geq q - (1 - q) \iff q \leq \frac{1 + x}{2}$$
$$U_1(O) \geq U_1(T) \iff x \geq -q + (1 - q) \iff q \geq \frac{1 - x}{2}\,.$$

Therefore, any strategy profile $((1,0,0),(q,1-q))$ with $q \in [{}^{1-x}/_{2}, {}^{1+x}/_{2}]$ is a NE.

All these equilibria are Trembling Hand perfect. Let $\sigma_1 = (1,0,0)$ and $\sigma_2 = (q, 1-q)$ with $q \in [{}^{1-x}/_{2}, {}^{1+x}/_{2}]$. As σ_2 is completely mixed, take $\sigma_2^k = \sigma_2$ for all $k \in \mathbb{N}$. Finally, take the sequence $(\varepsilon^k)_{k\in\mathbb{N}}$ with $\varepsilon^k > 0$, and let the candidate for Player 1's strategy be

$$\sigma_1^k = (1-\varepsilon^k, z\varepsilon^k, (1-z)\varepsilon^k) \to \sigma_1 .$$

For σ_2 to be a best response to σ_1^k we need

$$U_2(\sigma_1^k, H) = U_2(\sigma_1^k, T) \iff z\varepsilon^k - (1-z)\varepsilon^k = -z\varepsilon^k + (1-z)\varepsilon^k \iff z = \frac{1}{2} .$$

So σ_2 is a best response to σ_1^k when $z = 1/2$, and we found the required sequence. □

Exercise 5.5 (2019 Midterm)**.** Consider the variant of the Hawk-Dove game shown in Figure 5.3. (when $c > 1$ the game has the standard Hawk-Dove structure). Find of all Nash

	D	H
D	1,1	0,2
H	2,0	$1-c, 1-c$

Figure 5.3: The game in Exercise 5.5.

and trembling hand perfect equilibria for all values of c. Are the equilibrium strategies evolutionary stable?

Solution. It is clear that the value of c is relevant for a player when the other player is playign H. Moreover, H is best response if and only if $c \leq 1$. We analyze the three cases

1. $c < 1$. In this case H is strictly dominant, so (H, H) is the unique NE, which is trembling hand perfect, and H is an evolutionary stable strategy.
2. $c > 1$. In this case there are two pure strategy NE, (D, H) and (H, D) and a mixed strategy NE in which both players play D with probability $(c-1)/c$. The mixed strategy NE is clearly trembling hand perfect. Consider the sequence of strategies $(1-\varepsilon^k, \varepsilon^k) \to (1, 0)$. The action H is a best response to this each element of the sequence if

$$2(1-\varepsilon^k) + (1-c)\varepsilon^k \geq 1 - \varepsilon^k$$

which is clearly true if ε^k is small enough. Finally, take the sequence of strategies $(\varepsilon^k, 1-\varepsilon^k) \to (0, 1)$. The action D is a best response to each element of the sequence if

$$\varepsilon^k \geq 2\varepsilon + (1-c)(1-\varepsilon)$$

which again is true for ε^k small enough. Therefore the NE (D, H) and (H, D) are trembling hand perfect.
Our only candidate for a ESS is the mixed strategy $\alpha = ((c-1)/c, 1/c)$. As this is a completely mixed strategy equilibrium, it is clear that $u(\alpha, \alpha) = U(D, \alpha) = U(H, \alpha)$.

Hence α is an ESS if and only if $U(\alpha, D) > U(D, D)$ and $U(\alpha, H) > U(H, H)$. We have

$$U(\alpha, D) = \frac{c-1}{c} + 2\frac{1}{c} = \frac{1+c}{c} = 1 + \frac{1}{c} > 1 = U(D, D) > 1 - c = U(H, H)\,.$$

So α is an ESS.

3. $c = 1$: There are three NE: (D, H), (H, D), and (H, H). We check trembling hand perfection:
   - (H, H). As H is strictly better than D for Player i when Player $-i$ plays D, then adding tremble towards D for Player $-i$ does not make playing D for Player i more attractive. Therefore (H, H) is trembling hand perfect.
   - (D, H) (and (H, D)). Take the sequence of strategies $(\varepsilon^k, 1 - \varepsilon^k) \to (0, 1)$. The action D is a best response to each element of the sequence if $\varepsilon^k \geq 2\varepsilon$, which is obviously not true, so (D, H) and (H, D) are not trembling hand perfect.

   Finally, our only candidate for an ESS is H. Note that $U(H, H) = U(D, H)$ and $U(H, D) > U(D, D)$, so H is an ESS.

□

5.2 Extensive Games

Exercise 5.6. (2015 Midterm)

1. Find the sets of Nash and sequential equilibria of the game in Figure 5.4.
2. Give an example of an extensive-form game of perfect information with a subgame perfect equilibrium that is not trembling-hand perfect equilibrium.
3. Give an example of an extensive-form game of perfect information whose strategic form has trembling-hand perfect equilibrium that is not a subgame perfect equilibrium.

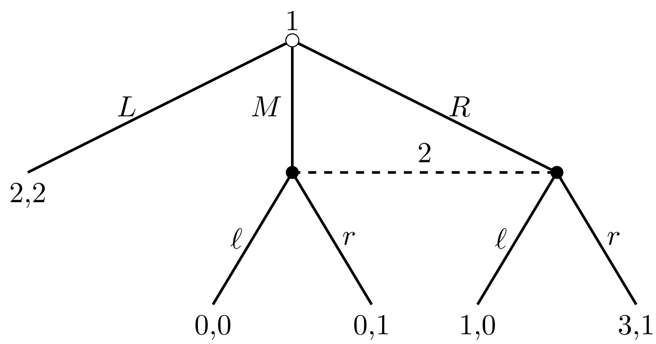


Figure 5.4: The game in Exercise 5.6.

Solution.

1. First, note that r is strictly dominant for Player 2. Given this, Player 1's best response is to play R. Finally, Player 2's beliefs when she has to play are $\mathbb{P}(h = (M)) = 0$ and $\mathbb{P}(h = (R)) = 1$. It is clear that this is the only equilibrium that satisfies sequential rationality.
2. The game shown in Figure 5.5 has such an equilibrium. Note that b is weakly dominant for Player 2, so Player 1 in indifferent between A and B in any SPE. Therefore $(B, (b, b))$ is a SPE. However, if Player 2 trembles it is strictly dominant for Player 1 to play A, so $(B, (b, b))$ is not trembling hand perfect.

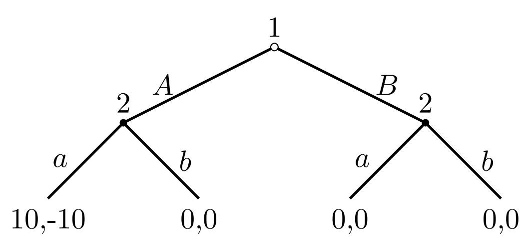


Figure 5.5: The strategy profile $(B, (b, b))$ is subgame perfect but not trembling hand perfect.

3. The game shown in Figure 5.6 is such an example. Note that $((A, D), a)$ is a NE and C is strictly dominant for Player 1 in the subgame following history (B, b). Hence $((A, D), a)$ is a NE that is not a SPE. However, for small trembles of Player 1, a is still the best response for Player 2, and for small trembles of Player 2, (A, D) is still a best response for Player 1. Therefore $((A, D), a)$ is trembling hand perfect.

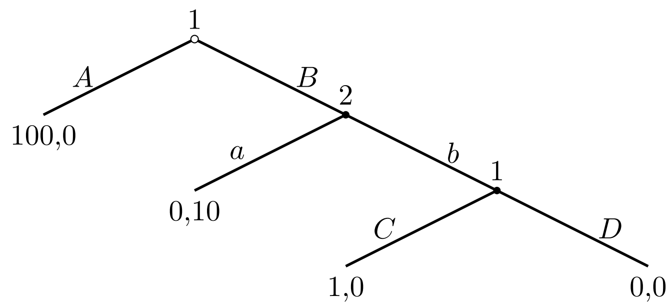
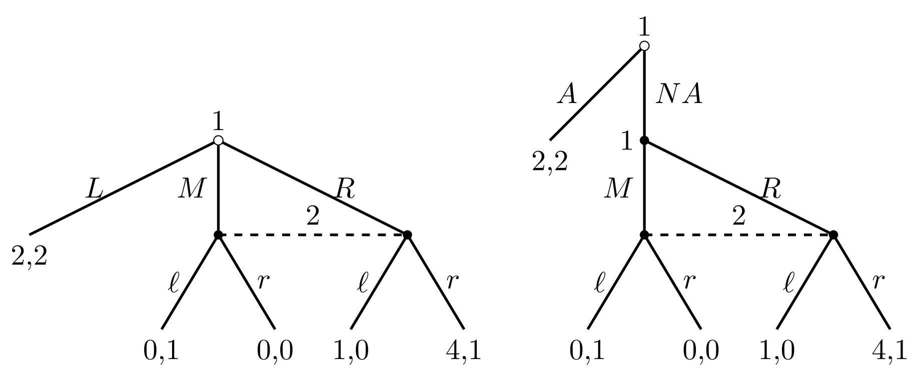


Figure 5.6: The strategy profile $((A, D), a)$ is trembling hand perfect but not subgame perfect.

□

Exercise 5.7 (2016 Midterm)**.** Find the sets of sequential equilibria of the two games in Figure 5.7 (Game II is obtained from Game I by adding a move to Player 1). Discuss the differences between the sets of equilibria in the two games.



(a) Game I

(b) Game II

Figure 5.7: The game in Exercise 5.7.

Solution. We analyze the games by backwards induction.

- Game I: Let μ be Player 2's beliefs of the history being (M) when her information set is reached. Then ℓ is weakly preferred to Player 2 if and only if $\mu \geq 1/2$. Also, let δ denote that probability of Player 2 playing ℓ. Then Player 1 weakly prefers L over R if and only if $\delta \geq 2/3$, and M is strictly dominated. Now we analyze sequential rationality for each possible value of μ.

1. $\mu > 1/2 \to$ P2 chooses ℓ, i.e. $\delta = 1 \to$ Player 1 chooses $L \to \mu$ cannot be computed using Bayes' rule.
2. $\mu < 1/2 \to$ P2 chooses r, i.e. $\delta = 0 \to$ Player 1 chooses $R \to$ by Bayes' rule $\mu = 0$, which is consistent with the initial condition $\mu < 1/2$.
3. $\mu = 1/2$: Let $\mathbb{P}(a)$ be the probability of Player 1 playing acton a. There are two cases that can induce $\mu = 1/2$.
   (a) $\mathbb{P}(M) = \mathbb{P}(R) > 0$: this is inconsistent with M being strictly dominated by R.
   (b) $\mathbb{P}(M) = \mathbb{P}(R) = 0$: in this case $\mathbb{P}(L) = 1$, which only happens if $\delta \geq 2/3$. This behavior of Player 2 is consistent with a belief $\mu = 1/2$.

We denote β_i Player i's behavioral strategy and $\beta = (\beta_1, \beta_2)$. We have three candidates for sequential equilibria. Let $(\varepsilon^k)_{k\in\mathbb{N}}$ be a sequence satisfying $\varepsilon^k > 0$ for all $k \in \mathbb{N}$ and $\varepsilon^k \to 0$. We check consistency.

1. $\beta = ((1,0,0),(1,0))$, $\mu > 1/2$. Take the sequence of profiles of behavioral strategies $(\beta^k)_{k\in N}$ defined by
$$\beta^k = \left(\left(1-\varepsilon^k-(\varepsilon^k)^2, \mu\varepsilon^k, (1-\mu)\varepsilon^k+(\varepsilon^k)^2\right), \left(1-\varepsilon^k, \varepsilon^k\right)\right) .$$
Note that strategies are completely mixed (the term $(\varepsilon^k)^2$ assures this in the case when $\mu = 1$). By Bayes' rule we have that
$$\mu^k = \frac{\mu\varepsilon^k}{\mu\varepsilon^k + (1-\mu)\varepsilon^k + (\varepsilon^k)^2} = \frac{\mu}{1+\varepsilon^k} \to \mu .$$
Therefore the assessment is consistent.
2. $\beta = ((0,0,1),(0,1))$, $\mu = 0$. As μ is directly revealed from Bayes' rule the assessment is consistent.
3. $\beta = ((0,0,1),(\delta, 1-\delta))$ for $\delta \geq 2/3$, $\mu = 1/2$. Take the sequence of profiles of behavioral strategies $(\beta^k)_{k\in N}$ defined by
$$\beta^k = \left(\left(1-2\varepsilon^k, \varepsilon^k, \varepsilon^k\right), \left(1-\varepsilon^k, \varepsilon^k\right)\right) .$$
By Bayes' rule we have that
$$\mu^k = \frac{\varepsilon^k}{\varepsilon^k + \varepsilon^k} = \frac{1}{2} \to \frac{1}{2} .$$
Therefore the assessment is consistent.

We have three types of sequential equilibria: (1) $\beta = ((0,0,1),(0,1))$, $\mu = 0$, (2) $\beta = ((0,0,1),(0,1))$, $\mu = 0$, and $\beta = ((0,0,1),(\delta, 1-\delta))$ for $\delta \geq 2/3$, $\mu = 1/2$.

- Game II: Let μ be the probability that Player 2 assigns to the history (NA, M) when she has to play (she assigns probability $1-\mu$ to (NA, R)). We know that Player 2 prefers ℓ if and only if $\mu \geq 1/2$. Conditional on the history (NA) it is strictly dominant for Player 1 to choose R, so every sequentially rational strategy has to prescribe for Player 1 to play R after (NA). Therefore by consistency we have that $\mu = 0$, and by sequential rationality Player 2 plays r. Finally, as Player 1 gets a utility of 4 in thie subgame after NA and gets 3 if she chooses A, she chooses NA at the beginning of the game.

  The only sequential equilibirum is $((0,0,1),(0,1))$, $\mu = 0$.

The difference between these two games is that for Player 2 to have a belief μ high enough in Game II Player 1 has to choose M with a (weakly) higher probability than R, and this never happens if Player 1 has to choose only between M and R. On the other hand, in Game I Player 1 might want to choose L, so she chooses only M and R by mistake. And as both are mistakes, it is possible for the mistake of choosing M to be more likely than the mistake of choosing R. $\square$

5.3 Repeated Games

Exercise 5.8 (2019 Midterm)**.** Consider the Prisoner's Dilemma game shown in Figure 5.8, where $y > x > 1$.

	C	D
C	x, x	$0, y$
D	$y, 0$	1,1

Figure 5.8: The Prisoner's Dilemma.

1. Find the condition on the discount factor $\delta \in (0,1)$ under which$(tit-for-tat, tit-for-tat)$ is a Nash equilibrium for the infinitely repeated game.
2. Show that $(tit-for-tat, tit-for-tat)$ is a subgame perfect equilibrium of the infinitely repeated game if and only if $\delta = x^{-1}$ and $y - x = 1$.

Solution. First, note that if both players play *tit-for-tat* the history is (C, C) in every period and the payoff is x. We analyze (wlog) deviations of Player 1 in the first period.

1. If Player 1 chooses D in the first period, the profile is (D, C) at $t = 1$, then Player 2 chooses D in at $t = 2$. Player 1 has two possible options for $t = 2$
   - C: In this case the profile is (C, D). Player 2 plays C at $t = 3$, and Plater 1 again has two options:
     - C. In this case the profile is (C, C)
     - D. In this case the profile is (D, C)

     Note that if choosing C at $t = 3$ is optimal for Player 1, then she prefers to induce the profile (C, C) over the profile (D, C). But then the initial deviation to (D, C) over (C, C) at $t = 1$ is not optimal.[1] Therefore, the deviation in which Player 1 plays C at $t = 2$ is maximized by alternating between D and C, which generates the history $(D, C), (C, D), (D, C), \ldots$. the value for Player 1 of this history is

     $$(1-\delta)\left[y + \delta^2 y + \delta^4 y + \ldots\right] = (1-\delta)\frac{y}{1-\delta^2} = \frac{y}{1+\delta}\,.$$

   - D: In this case the profile is (D, D). Player 2 plays D at $t = 3$ and Player 1 has two options.
     - C: In this case the profile is (C, D).
     - D: In this case the profile is (D, D).

[1] This analysis is true here *only* because the actions that Player 2 chooses depend only on what Player 1 did in the previous period.

Note that if Player 1 chooses C at $t = 3$, then she prefers the profile (C, D) over the profile (D, D). But then she would also have preferred this at $t = 2$, which is inconsistent with the choice D at $t = 2$. then for this deviation to be more attractive than choosing C at $t = 2$ we need for (D, D) to be more attractive (for Player 1) than (C, D). therefore it induces the history $(D, C), (D, D), (D, D), (D, D), \ldots$, whose payoff is

$$(1-\delta)\left[y + \delta + \delta^2 + \ldots\right] = (1-\delta)y + \delta = y - \delta(y-1)\,.$$

Finally (*tit-for-tat*,*tit-for-tat*) is a NE if none of these two strategies is a profitable deviation. This is

$$\left(x \geq \frac{y}{1+\delta} \iff \delta \geq \frac{y-x}{x}\right)\ ;\ \text{and}\ \left(x \geq y - \delta(y-1) \iff \delta \geq \frac{y-x}{y-1}\right)\,.$$

2. Note that after the strategy profile (*tit-for-tat*,*tit-for-tat*) the only part of the history that influences what happens in the future is what happened in the last period. Therefore there are 5 sets of histories that we need to analyze. First we look at the histories they induce in the future under (*tit-for-tat*,*tit-for-tat*).
   - ϕ: In this case the induced history is $h_1 \equiv ((C, C), (C, C), \ldots)$. The value of h_1 for Player 1 is $V_1(h_1) = x$.
   - History ends in (C, C): In this case the induced history is also h_1.
   - History ends in (C, D): In this case the induced history is $h_2 \equiv ((D, C), (C, D), (D, C), \ldots)$, whose value is $V_1(h_2) = y/(1+\delta)$.
   - History ends in (D, C): In this case the induced history is $h_3 \equiv ((C, D), (D, C), (C, D), \ldots)$, whose value is $V_1(h_3) = \delta V_1(h_2) = \delta y/(1+\delta)$.
   - History ends in (D, D): In this case the induced history is $h_4 \equiv ((D, D), (D, D), (D, D), \ldots)$, whose value is $V_1(h_4) = 1$.

   By the one deviation property we analyze strategies in which Player 1 deviates for one period and the continue playing *tit-for-tat*.
   - If deviates after ϕ, induces history h_2. There's no profitable deviation if $V_1(h_1) \geq V_1(h_2)$.
   - If deviates after (C, C), induces history h_2. There's no profitable deviation if $V_1(h_1) \geq V_1(h_2)$.
   - If deviates after (C, D), induces history h_1. There's no profitable deviation if $V_1(h_2) \geq V_1(h_1)$.
   - If deviates after (D, C), induces history h_4. There's no profitable deviation if $V_1(h_3) \geq V_1(h_4)$.
   - If deviates after (D, D), induces history h_3. There's no profitable deviation if $V_1(h_4) \geq V_1(h_3)$.

All these conditions imply $V_1(h_1) = V_1(h_2)$ and $V_1(h_3) = V_1(h_4)$, from where it is straightforward to derived the required conditions.

$\square$

Exercise 5.9 (Problem Set 6)**.** Find the conditions on x, y, and δ such that the following strategy is a Nash Equilibrium: "Choose C in period 1 and after any history in which the outcome of the last period is either (C, C) or (D, D); choose D after any other history." (That is, choose the same action again if the outcome was relatively good for you, and switch actions if it was not.)

Solution. The best deviation for Player 2 leads to the sequence of outcomes that alternates between (C, D) and (D, D). The discounted average payoff of this sequence of outcomes is

$$\begin{aligned}(1-\delta)\left[y+\delta+y\delta^2+\delta^3+\dots\right] &= (1-\delta)\left[y+y\delta^2+\dots\right]+(1-\delta)\left[\delta+\delta^3+\dots\right]\\ &= (1-\delta)\frac{y+\delta}{1-\delta^2}\\ &= \frac{y+\delta}{1+\delta}\,.\end{aligned}$$

On the other hand the discounted average of the constant sequence containing only (C, C) is x. Thus for the strategy pair to be a Nash equilibrium we need

$$x \geq \frac{y+\delta}{1+\delta} \iff \delta \geq \frac{y-x}{x-1}\,.$$

$\square$

5.4 Bargaining

Exercise 5.10 (2019 Midterm)**.** Show that in the standard Nash bargaining problem WPO can be replaced with SIR.

- Weak Pareto Efficiency (WPO): If $\langle S, d\rangle$ is a bargaining problem where $s, \in S$, $t \in S$ and $t_i > s_i$ for $i = 1, 2$, then $f(S, d) \neq s$.
- Strict Individual Rationality (SIR): In any bargaining problem $\langle S, d\rangle$ we have $f(S, d) \gg d$.

That is, the Nash bargaining solution

$$f^N(S,d) = \arg\max_{\substack{s\in S \\ s\geq d}} (s_1 - d_1)(s_2 - d_2)$$

is the only solution satisfying SYM, SIR, INV and IIA (it is sufficient to show that these four axioms are equivalent to the standard axioms SYM, WPO, INV and IIA).

Solution. Wlog we set $d = 0$, and write $f(S, d) = f(S)$. Also, denote $AX = SYM + INV + IIA$. We prove both directions:

- ($AX + WPO \implies AX + SIR$): Suppose f satisfies SYM, WPO, INV and IIA, then we know that $f = f^N$. It is clear that f^N satisfies SIR.
- ($AX+SIR \implies AX+WPO$): Suppose f satisfies SYM, SIR, INV and IIA, and let $z = f(S)$. By SIR we have $z \gg 0$. Towards a contradiction, suppose z is not weakly Pareto Optimal. Then exists $s \in S$ such that $s_1 \geq z_1$, $s_2 \geq z_2$, and $s \neq z$. let $a_i = {}^{z_i}/_{s_i}$. Then $a_i \leq 1$ and $a = (a_1, a_2) \neq (1, 1)$. Define $S' = \{(a_1 s_1, a_2 s_2) : (s_1, s_2) \in S\}$. We have that (1) $S' \subset S$, and (2) $z \in S'$. Then by IIA we have that $f(S') = z$. But as $a \neq (1, 1)$ we have that $f(S') = z$ contradicts INV. Therefore f has to satisfy WPO.

$\square$

Exercise 5.11 (2017 Midterm)**.** Consider the set of all bargaining problems $\langle S, d\rangle$ where $d \in S$ and S is compact, convex, and comprehensive and and there exists $s \in S$ such that $s \gg d$.

The proportional bargaining solution f^P can be defined in terms of the social welfare function

$$f^P(S,d) = \arg\max_{\substack{s\in S \\ s\geq d}} \min\{\alpha(s_1 - d_1), \beta(s_2 - d_2)\}\,;$$

where $\alpha, \beta > 0$.

Determine whether the proportional bargaining solution satisfies each of the axioms of Nash invariance to equivalent utility representations, symmetry, independence of irrelevant alternatives, and weak Pareto efficiency.

Solution. First, note that the solution is not unique. Take for example $\alpha = \beta = 1$ and

$$S = co(\{(0,0),(1,0),(1,2),(0,2)\}),$$

where co refers to the convex hull. Note that both $(1,1)$ and $(1,2)$ are maximizers.

- f^P satisfies IIA. Suppose $S' \subset S$, $z \in f^P(S)$ and $z \in S'$. Towards a contradiction suppose $z \notin f^P(S')$. Then there is $z' \in S'$ such that

  $$\min\{\alpha(z'_1 - d_1), \beta(z'_2 - d_2)\} > \min\{\alpha(z_1 - d_1), \beta(z_2 - d_2)\}.$$

  But as $S' \subset S$ we have that $z' \in S$, which contradicts the fact that $z \in f^P(S)$.
- f^P does not satisfy INV. Take $\alpha = 1$, $\beta = 2$, and $S = co(\{(0,0),(1,0),(0,1)\})$. Then $f^P(S) = (2/3, 1/3)$. Now take the set $S' = \{(2s_1, s_2) : (s_1, s_2) \in S\}$. Then $f^P(S') = (1, 1/2)$. As $f_2^P(S) \neq f_2^P(S')$ we have that INV is violated.
- f^P satisfies WPO. Suppose $s, t \in S$ and $t \gg s$ (i.e., $t_1 > s_1$ and $t_2 > s_2$). Then

  $$\min\{\alpha(t_1 - d_1), \beta(t_2 - d_2)\} > \min\{\alpha(s_1 - d_1), \beta(s_2 - d_2)\}.$$

  Therefore $s \notin f^P(S)$.
- f^P does not satisfy SYM in general. Take $\alpha \neq \beta$, and $S = co(\{(0,0),(1,0),(0,1)\})$. Then S is symmetric but

  $$f_1^P(S) = \frac{\beta}{\alpha + \beta} \neq \frac{\alpha}{\alpha + \beta} = f_2^P(S).$$

□

Exercise 5.12 (2016 Midterm)**.** Let $\mathbf{B}$ be the set of all convex, compact and comprehensive sets in $\mathbb{R}^2_+$ with nonempty intersection with $\mathbb{R}^2_{++}$.

1. Show that the Kalai bargaining solution

$$f^K(S,d) = \{s \in S : s_1 = s_2\} \cap WPO(S)$$

   does not satisfy INV.

2. Show that the Kalai-Smorodinsky bargaining solution

$$f^K(S,d) = \left\{s \in S : \frac{s_1}{\bar{s}_1} = \frac{s_2}{\bar{s}_2}\right\} \cap WPO(S)\,,$$

   with $\bar{s}_i = \arg\max_{s \in S} s_i$, does not satisfy IIA.

Solution.

1. Take $S = co(\{(0,0),(1,0),(0,1)\})$. Then $f^K(S) = (1/2, 1/2)$. Now take $S' = \{(2s_1, s_2) : (s_1, s_2) \in S\}$. Then $f^K(S') = (2/3, 2/3) \neq (1, 1/2)$.
2. Take $S = co(\{(0,0),(1,0),(0,1)\})$. Then $\bar{s}_1 = \bar{s}_2 = 1$ and $f^{KS}(S) = (1/2, 1/2)$. Now take $S' = co(\{(0,0),(1/2,0),(1/2,1/2),(0,1)\})$. Then $\bar{s}'_1 = 1/2$, $\bar{s}'_2 = 1$, and $f^{KS}(S') = (1/3, 2/3)$. We have that $f^{KS}(S) \in S'$, $S' \subset S$, and $f^{KS}(S) \neq f^{KS}(S')$.

$\square$

Exercise 5.13 (2020 Midterm)**.** Let $\mathcal{B}$ denote the set of all bargaining problems $\langle S, d\rangle$ where $d \in S$ and S is compact and convex. The *linear* bargaining solution can be defined in terms of the social welfare function

$$\alpha(s_1 - d_1) + \beta(s_2 - d_2)$$

where $\alpha, \beta > 0$. (When $\alpha = \beta$ the proportional bargaining solution leads to exactly the same outcome as the utilitarian criterion.) Determine whether the proportional bargaining solution satisfies each of the axioms of Nash – SYM, WPO, INV and IIA.

Solution. Let $f^L(S, d)$ be the solution. Define the problem $\langle \bar{S}, \bar{d}\rangle$ by $\bar{d} = (0, 0)$ and

$$\bar{S} = \{(s_1, s_2) \in \mathbb{R} : s_1 + s_2 \leq 1\}.$$

- SYM: Take $\alpha > \beta$. Note that $\langle \bar{S}, \bar{d}\rangle$ is a symmetric problem. However the solution in this case is $s^\star = f^L(\bar{S}, \bar{d}) = (1, 0)$, which does not satisfy $s_1^\star = s_2^\star$. Therefore f^L does not satisfy SYM.
- WPO: Suppose $s, t \in S$, where $s_i > t_i$ for $i = 1, 2$. Then as $\alpha, \beta > 0$ we have that

$$\alpha(s_1 - d_1) + \beta(s_2 - d_2) > \alpha(t_1 - d_1) + \beta(t_2 - d_2)$$

  and t cannot be the solution to $\langle S, d\rangle$. Therefore f^L satisfies WPO.
- INV: Take $\alpha = 1$ and $\beta \in (1, 2)$. We have that $s^\star = f^L(\bar{S}, \bar{d}) = (0, 1)$. Now take

$$S' = \{(2\,s_1, s_2) : (s_1, s_2) \in \bar{S}\}.$$

  Note that S' is obtained from $\bar{S}$ by the transformations $s_i \mapsto \alpha_i s_i + \beta_i$ with $\alpha_1 = 2$, $\alpha_2 = 1$, and $\beta_1 = \beta_2 = 0$. We have that $s^{\star\star} = f^L(S', \bar{d}) = (2, 0)$. Therefore $s_i^{\star\star} \neq \alpha_i s_i^\star + \beta_i$ for $i = 1, 2$, and f^L does not satisfy INV.
- IIA. Take two bargaining problems $\langle S, d\rangle$ and $\langle S', d\rangle$ with $S' \subset S$, and assume $s^\star = f^L(S, d) \in S'$. By definition of f^L for every $s \in S$ we have[2]

$$\alpha(s_1^\star - d_1) + \beta(s_2^\star - d_2) > \alpha(s_1 - d_1) + \beta(s_2 - d_2).$$

  As $S' \subset S$ we have that the previous condition holds for every $s' \in S'$. Therefore $s^\star = f^L(S', d)$, and f^L satisfies IIA.

□

[2] Here we assume the solution is unique. To extend to the general case requires just to replace $s^\star = f^L(S, d)$ by $s^\star \in f^L(S, d)$, and the strict inequality condition by a weak inequality.