Smooth Rationalization

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Abstract

Economic models usually endow agents with (well-behaved) differentiable utilities. However, the behavioral implications of such an assumption are unclear. We study conditions under which choices can be rationalized by a differentiable utility, i.e., smoothly rationalized. Differentiability implies that indifferent choices have the same marginal rate of substitution. Starting from this observation, we develop an exact test for smooth rationalization. We also show that the existence of higher-order derivatives, commonly used for comparative statics, is empirically costless. We test smooth rationalization into several experimental data sets and find that, in most cases, choices are consistent with a differentiable utility.

EconLit Codes: D01 (Microeconomic Behavior: Underlying Principles), D11 (Consumer Economics: Theory), D12 (Consumer Economics: Empirical Analysis)

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The axioms of the theory must be formulated in terms of observable choices made by a consumer among commodity vectors

Gérard Debreu (1972, p.605)

Differentiability of the utility function is one of the more elusive assumptions in economics. It is a widespread and helpful assumption as it allows to characterize optimal choices through first-order conditions, significantly simplifying the mathematical analysis and adding tractability to economic models, comparative statics, and their applications. However, there is no clear interpretation of the behavioral implications of assuming a differentiable utility function. Unlike other assumptions, such as monotonicity, continuity, or concavity, the axioms for a differentiable utility function are motivated by geometry, not behavior. When discussing a differentiable utility Deaton and Muellbauer (1980) assert that

> Nothing in our axioms will guarantee this nor can new axioms be introduced to do so without making patently unrealistic assumptions.

The study of individual decisions in economics can be, in broad terms, divided into two main branches. The first one, usually referred to as decision theory, assumes the observation of an infinite amount of choices in a way that reveals the complete preference relation of the agent. The second branch, usually referred to as revealed preference theory, assumes the observation of only a finite amount of data.[1] In this paper, we study the assumption of a differentiable utility within the revealed preferences tradition. Our main result presents necessary and sufficient conditions for rationalizing consumer choice data by a well-behaved (i.e., strictly increasing, continuous, and concave) and differentiable utility function. We call this property *smooth rationalization*.

There are two different but related benefits of providing a characterization of smooth rationalization. First, it provides a theoretical understanding of the restrictions imposed by

[1] Chapter 0.2 in Chambers and Echenique (2016) present an excellent discussion of the distinction between decision and revealed preference theory. The term revealed preference to refer to the empirical content of utility analysis is coined by Samuelson (1947).

a differentiable utility on the behavior of a price-taker consumer, i.e., in the classical consumer problem. Given that the impositions of these assumptions over preference relations are not well understood, this exercise is particularly relevant. Second, our conditions can be used to test for a differentiable utility in actual data. Because there is a vast applied economics literature that relies on first-order conditions to characterize results, testing for differentiability of the utility function is a critical exercise in understanding the validity of the conclusions of this literature.

The revealed preference literature that starts with Samuelson (1938). The Afriat Theorem, the seminal result in this literature (Afriat, 1967; Varian, 1982), states that choices can be rationalized by a well-behaved utility if and only if they satisfy the Generalized Axiom of Revealed Preferences (GARP). Focusing on differentiability and strict convexity, Chiappori and Rochet (1987) show that an invertible demand, a condition they call the Strong Version of the Strong Axiom of Revealed Preferences (SSARP) is sufficient to achieve a differentiable utility function. We develop examples showing that, in general, an invertible demand is neither necessary nor sufficient for a differentiable utility.

We implement our test on experimental data to assess the empirical restrictiveness of assuming a differentiable utility. The summary of the results is presented in Table 1. Contrary to the suggestion of Deaton and Muellbauer (1980) that differentiability requires "patently unrealistic assumptions," a vast majority (92%) of subjects present choices consistent with a differentiable utility, i.e., are smoothly rationalizable. This result is robust when we relax the rationalization requirements using the Houtman and Maks (1985) Index. Furthermore, we compare our characterization with the (partial) test developed by Chiappori and Rochet (1987) and find a big difference: 83% of the subjects who fail SSARP are false negatives, i.e., have smoothly rationalizable choices. These results show the empirical relevance of our test, as using SSARP as a test for differentiability would incorrectly conclude that a differentiable utility is inconsistent with observed choices for almost half of the subjects.

The starting tool to test for smooth rationalization is the idea of *revealed indifferences*, which arise if two different choices are revealed preferred to each other. Our first observation is that whenever two choices are revealed indifferent to each other, concavity of the utility function implies that the indifference set between such choices must be flat. Hence, both

Table 1: Aggregated Results

N	$\frac{\text{Smooth}}{\text{GARP}}$	$\frac{\text{SSARP}}{\text{GARP}}$	$\frac{\text{Smooth - SSARP}}{\text{GARP - SSARP}}$
4958	91.9%	51.7%	83.3%

N is the number of subjects; GARP is the number of rationalizable subjects; Smooth is the number of smoothly rationalizable subjects; SSARP is the number of subjects who satisfy this axiom.

choices have the same marginal rate of substitution (MRS). Furthermore, we show that when several observed choices are revealed indifferent to one another, each choice is optimal from not only its own price but also the meet of all the prices at which the indifferent choices were made.[2] This meet of prices provides tighter bounds on the different MRSs at any chosen bundle. Motivated by this observation, we propose a modification of the data set that replaces the original price with the meet among the prices of choices that reveal indifferent to each other. Our first result shows that smooth rationalization of the original data set and its modification are equivalent. Our second result shows that differentiability imposes no additional behavioral restrictions beyond the tighter bounds on the MRSs. If a data set is invariant to the modification, then rationalization and smooth rationalization are equivalent.

The two previous results allow us to develop a simple exact test for smooth rationalization. First, starting from the original data set, we apply the modification until a fixed point is reached (which is assured to happen after a finite number of steps). Since this fixed point is invariant to the modification, it is smoothly rationalizable if, and only if, it is rationalizable, i.e., satisfies GARP. Furthermore, since smooth rationalization of the modified and of the original data set are equivalent, smooth rationalization of the original data set and of the fixed point of the modification are also equivalent. Therefore, the original data set is smoothly rationalizable if, and only if, the fixed point of the modification satisfies GARP.

Our characterization of smooth rationalization also shows that whenever choices are

[2]For a given set of K-dimension vectors, the meet of such vectors is the vector resulting from taking the coordinate-wise minimum among the set.

smoothly rationalizable, we can do such rationalization using an infinitely differentiable utility. Hence, to assume the existence of second- and higher-order derivatives of the utility function comes at no cost from an empirical perspective. The existence of higher-order derivatives, especially second-order ones, is a valuable resource for performing comparative statics, for example, through the Implicit Function Theorem or Taylor approximations. Finally, we develop an additional characterization through (a modified version of) the Afriat inequalities: a set of linear inequalities that have a solution if, and only if, the data set is smoothly rationalizable. Our inequalities have a direct interpretation through the first-order conditions of the utility maximization problem.

Intuitively, a differentiable utility is achieved by constructing smooth indifference sets, hence the focus on bundles that reveal indifferent to each other. The Online Appendix shows that the same approach can be applied to achieve smooth rationalization when we require more structure on the utility function. Specifically, we study smooth rationalization by a strictly concave utility, by a quasilinear utility, and by a homothetic utility. The main difference in each case is that the specific structure allows us to infer a different set of indifferences between bundles.

Within the decision theory literature, i.e., assuming a complete observation of the consumer's preference relation, Debreu (1972) characterizes the differentiability of the utility function. In the words of Neilson (1991), differentiability "requires that indifference sets be smooth and vary smoothly"; that is, indifference sets do not have kinks, and the change in utility level must also be smooth. Our characterization of a smooth utility function rationalizing demand data implies that only the first condition in Debreu (1972), smooth indifference sets, is relevant under finite data. In contrast, smooth variation of the utility level cannot be tested.

Related Literature

Our work contributes to the revealed preference literature. The problem of rationalizing a data set of choices starts with Samuelson (1938). The seminal result of this literature is the Afriat Theorem (Afriat, 1967; Diewert, 1973; Varian, 1982), which has led to a vast literature on rationalizing observed choices. Fostel et al. (2004); Polisson and Renou (2016);

Beggs (2021) study new proofs of the Afriat Theorem that highlight different aspects and interpretations of the result.

Matzkin and Richter (1991) studies the problem of rationalization by a strictly concave utility, which generates a single-valued demand, and show that the Strong Axiom of Revealed Preferences (SARP; Houthakker, 1950) is necessary and sufficient for such rationalization. Taking a further step, Lee and Wong (2005) show that under SARP, the rationalizing utility can be chosen to generate an infinitely differentiable demand. Chiappori and Rochet (1987) study the problem of smooth rationalization under SARP and provide SSARP as a sufficient condition. The paper by Chiappori and Rochet (1987) is the closest to ours regarding the research question and the methodology.

Besides Matzkin and Richter (1991), a vast literature studies rationalization by utility functions with further structure. Two examples are Varian (1983) for homothetic utilities and Brown and Calsamiglia (2007) for quasilinear utilities. In the Online Appendix, we study smooth rationalization for the three cases mentioned above.

Within the decision theory literature, i.e., assuming infinite data, the seminal characterization of a differentiable utility function is provided by Debreu (1972). He shows that continuous and increasing preferences have a representation by a differentiable utility if, and only if, the set of indifference sets is a smooth manifold. Alternatively, Rubinstein (2012) proposes a property of differentiable preferences that can be interpreted as the existence of a marginal utility vector. Diasakos and Gerasimou (2022) show that Rubinstein (2012) differentiable preferences are equivalent to the utility function having smooth indifference sets, which is weaker than the function being differentiable (see Neilson, 1991).

Finally, our empirical analysis relies on the experimental design pioneered by Choi et al. (2007b). Several tools are proposed to recover preferences in the case that a subject fails GARP; we choose the one proposed by Houtman and Maks (1985) because it is the only one among the most popular ones (the others two being Afriat, 1973; Varian, 1990) that allows us to differentiate between differentiable and nondifferentiable utilities (Ugarte, 2023b).

The remainder of the paper proceeds as follows: Section 1 introduces the problem and presents the main definitions and previous results relevant to our analysis. Section 2 develops the modification of the data set that is our primary tool to develop our results.

Section 3 presents our main result: an exact test for smooth rationalization. Section 4 implements our result into experimental data. Finally, Section 5 concludes. Proofs are provided in the appendices.

1 Rationalization and Invertible Demand

1.1 Preliminaries

Our analysis starts from an agent's demand data set, comprised of N different observations. The agent consumes bundles of K nonnegative goods, i.e., the consumption space is $\mathbb{R}^K_+$.[3] In each observation $i \in [N]$, the agent faces a strictly positive price vector $p^i \in \mathbb{R}^K_{++}$ and spends an amount of money normalized to one without loss of generality. Given p^i, the agent chooses a bundle x^i from the budget set $\{x \in \mathbb{R}^K_+ : p^i \cdot x \leq 1\}$. Together, prices and chosen bundles form the data set $\mathcal{D} = (x^i, p^i)_{i\in[N]}$, which is the primitive of our problem. We refer to rounds $i \in [N]$ as observations, and to bundles x^i as chosen bundles or choices. As standard in the revealed preference literature (and unavoidable for rationalization by any meaningful utility), we assume that the agent spends all her income on each choice, i.e., $p^i \cdot x^i = 1$.

For two bundles x and y, we can infer that x is preferred to y if when x was chosen, y was also available. Moreover, the idea of monotonicity, that "more is better", leads us to infer that if x was chosen and y was not only available but also strictly cheaper, then x has to be strictly preferred to y. Along with the idea of transitivity, the previous notion leads to the classic definition of revealed preferences.

Definition 1. *For a data set $\mathcal{D}$ and two choices x^i and x^j, x^i is*

- directly revealed preferred to x^j *(denoted $x^i \succsim^\star x^j$) if $p^i \cdot x^j \leq 1$;*

[3] We work with the following notation. $\mathbb{N}$ is the set of natural numbers and $\mathbb{R}$ the set of real numbers. $\mathbb{R}_+$ is the set of positive numbers including zero, and $\mathbb{R}_{++} = \mathbb{R}_+ \setminus 0$ excludes it. For any $M \in \mathbb{N}$ we set $[M] = \{1, 2, \ldots, M\}$. A vector $x \in \mathbb{R}^M$ is $x = (x_1, x_2, \ldots, x_M)$, and $||x||$ is its Euclidean norm. The zero vector is $\mathbf{0}$, with dimensionality implied by the context. For any two vectors $x, y \in \mathbb{R}^M$ we write $x \geq [\gg] y$ if $x_i \geq [>] y_i$ for all $i \in [M]$, $x > y$ if $x \geq y$ and $x \neq y$, and $x \cdot y$ for its dot product. A function $f : \mathbb{R}^M \to \mathbb{R}$ is strictly increasing if $x > y$ implies $f(x) > f(y)$; if f is differentiable, $\nabla f(x)$ is its gradient at x. $\mathbb{I}_{\{P\}}$ is the indicator function, which takes value 1 if P is true, and zero otherwise.

- directly revealed strictly preferred to x^j *($x^i \succ^\star x$) if $p^i \cdot x^j < 1$;*
- revealed preferred to x^j *($x^i \succsim x^j$) if there is a sequence of observations $(m_\ell)_{\ell \in [L]}$ such that $x^i \succsim^\star x^{m_1} \succsim^\star x^{m_2} \succsim^\star \ldots \succsim^\star x^{m_L} \succsim^\star x^j$; and*
- revealed indifferent to x^j *($x^i \sim x^j$) if $x^i \succsim x^j$ and $x^j \succsim x^i$.*

Since the relation of revealed indifference is an equivalence relation, whenever $x^i \sim x^j$ we also say that x^i and x^j are revealed indifferent to each other. We state that x^i is not directly revealed preferred to x^j by $x^i \not\succsim^\star x^j$, and analogously for the other relations.

Afriat (1967) uses revealed preferences to show that we can interpret the choices in $\mathcal{D}$ as being driven by a locally nonsatiated utility function (this is, there is a locally nonsatiated U satisfying $U(x^i) \geq U(x)$ whenever $p^i \cdot x \leq 1$) if, and only if, it satisfies a condition he called cyclical consistency.[4] Furthermore, the rationalizing utility can always be chosen to be strictly increasing, continuous, and concave.[5] Given the theoretical soundness of these properties, we will refer to utilities that are strictly increasing, continuous, and concave, as *well-behaved* utilities. The most popular version of cyclical monotonicity is the Generalized Axiom of Revealed Preferences, proposed by Varian (1982).

Definition 2. *$\mathcal{D}$ satisfies the* Generalized Axiom of Revealed Preferences*,* GARP*, if for any pair of choices x^i and x^j, $x^i \succsim x^j$ implies $x^j \not\succ^\star x^i$*

Usually, applied economic research imposes more structure on the utility function than the one obtained in the Afriat Theorem. The two more common additional assumptions are strict concavity and differentiability. A strictly concave utility has a unique optimal choice from each price vector, which implies that the induced demand is a single-valued function. Differentiability implies that optimal choices of the utility maximization problem can be characterized by first-order conditions, which make them tractable to perform comparative statics. Matzkin and Richter (1991) show that the Strong Axiom of Revealed Preferences (SARP, Houthakker, 1950) is necessary and sufficient to rationalize a data set by a strictly increasing, continuous, and strictly concave utility.

[4] U is *locally nonsatiated* if for every $x \in \mathbb{R}^K_+$ and $\varepsilon > 0$ there is $y \in \mathbb{R}^K_+$ s.t. $||x-y|| < \varepsilon$ and $U(y) > U(x)$.

[5] Forges and Minelli (2009) and Cherchye et al. (2014) discuss how to test concavity for non-linear budget sets.

Definition 3. *$\mathcal{D}$ satisfies the* Strong Axiom of Revealed Preferences, SARP*, if for any two different choices $x^i \neq x^j$, $x^i \succsim x^j$ implies $x^j \not\succsim^\star x^i$.*

The relation between SARP and a single-valued demand can be understood by noticing that SARP holds if, and only if, GARP holds and $x^i \neq x^j$ implies $x^i \not\sim x^j$.

We study the empirical content of the utility function's differentiability assumption. We know that any concave utility function is generically differentiable, i.e., differentiable in a dense subset of its domain (Rockafellar, 2015, Theorem 25.5). However, that does not translate into a zero probability of observing choices in the nondifferentiable portion of such utility. An extreme example of such a case is the Leontief utility $U(x) = \min_{k\in[K]} x_k$. For any price p, the optimal choice of the Leontief utility satisfies $x_1 = x_2 = \ldots = x_K$. Hence, all choices lay in the nondifferentiable portion of the function. Although this utility is generically differentiable, the first-order conditions never characterize the utility-maximizing bundle.

Since differentiability is usually defined in open sets, but our consumption space $\mathbb{R}^K_+$ is closed, we use a definition of differentiability that allows for a well-defined derivative at the boundary.

Definition 4. *Take $X \subset \mathbb{R}^K$. A function $f : X \to \mathbb{R}$ is* differentiable *if there is an open set Y such that $X \subset Y$ and a differentiable function $g : Y \to \mathbb{R}$ that extends f, i.e., that satisfies $f(x) = g(x)$ for every $x \in X$.*

Following the Afriat Theorem, we focus on rationalization by well-behaved utilities.

Definition 5. *$\mathcal{D}$ is* smoothly rationalizable *if there is a differentiable, strictly increasing, and concave utility function $U : \mathbb{R}^K_+ \to \mathbb{R}$ such that, for every $i \in [N]$, $U(x^i) \geq U(x)$ whenever $p^i \cdot x \leq 1$. Such function* smoothly rationalizes *$\mathcal{D}$.*

1.2 Differentiability and Invertible Demand

Focusing on single-valued demands, i.e., on SARP, Chiappori and Rochet (1987) study conditions under which the data can be smoothly rationalized. Their motivation is the data set in panel (a) of Figure 1, in which two different budget sets have the same optimal choice. The reason such data is not smoothly rationalizable is explicit: the indifference curve going through the bundle x^t has to have a kink at x^t. If not, it would go to the interior of at least one of the budget sets, which is impossible if choices are optimal. Since the indifference curve has a kink, the utility function cannot be differentiable. In terms of the first-order conditions, a differentiable utility rationalizing the data implies that, in each choice, the ratio of prices has to be equal to the marginal rate of substitution (MRS). This condition fails if the same bundle is chosen from two different price vectors since the expenditure normalization requires that two prices have (some) different price ratios.[6]

To avoid situations like the one in panel (a) of Figure 1, Chiappori and Rochet (1987) require for the observed demand to be invertible; this is, for different prices to have different choices. They call this condition Strong SARP.

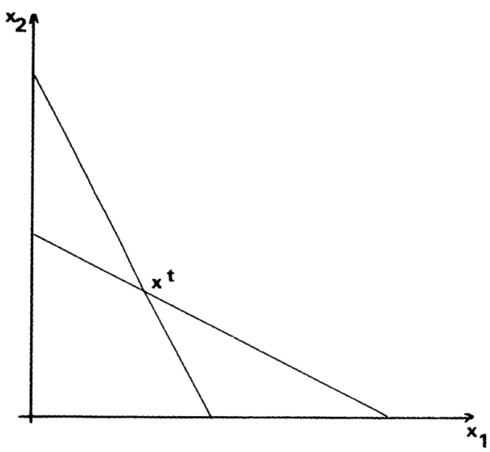
Definition 6 (Chiappori and Rochet (1987))**.** *$\mathcal{D}$ satisfies the* Strong version of SARP *(SSARP) if it satisfies SARP and for all observations i, j, $p^i \neq p^j$ implies $x^i \neq x^j$.*

The main result in Chiappori and Rochet (1987) is the sufficiency of SSARP for smooth rationalization by a strictly concave utility. Furthermore, they show that the rationalizing utility can be chosen to be infinitely differentiable.[7] Although the authors do not claim the necessity of SSARP for smooth rationalization under a strictly concave utility, this condition is usually referred to in the literature as necessary and sufficient, a mistake we track back to Theorem 1^∞ in Matzkin and Richter (1991). Panel (b) in Figure 2, explained in the paragraph below, shows a counterexample to the necessity of SSARP.

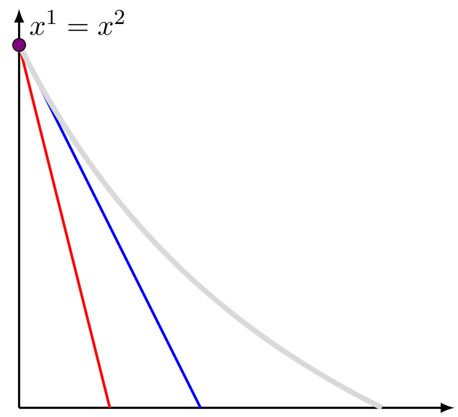
Despite the sufficiency of SSARP, an invertible and rationalizable data set is neither necessary nor sufficient for smooth rationalization. Panel (b) of Figure 2 presents a data set that satisfies SARP, is not invertible, but is smoothly rationalizable; the reason why SSARP is not necessary in this case is that equality between MRS and price ratio is not needed for corner solutions. Panels (c) and (d) show data sets that fail SARP, i.e., present indifferences between choices. The data set in (c) is invertible and smoothly rationalizable. It illustrates an important point for our result: choices that are indifferent to each other must have a

[6] Without expenditure normalization, the requirement would be for prices and expenditures not to be proportional to each other.

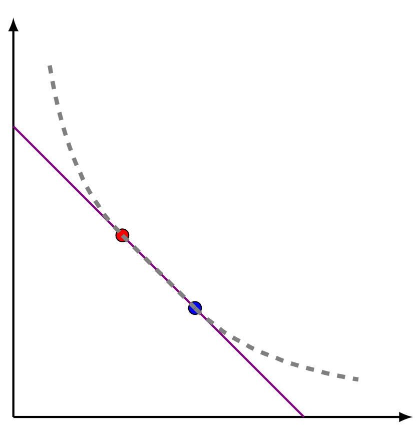
[7] Specifically, Chiappori and Rochet (1987) show the result for rationalization on a compact subset of $\mathbb{R}^K_+$. Matzkin and Richter (1991, Theorem 1^∞) show that such restriction is unnecessary.



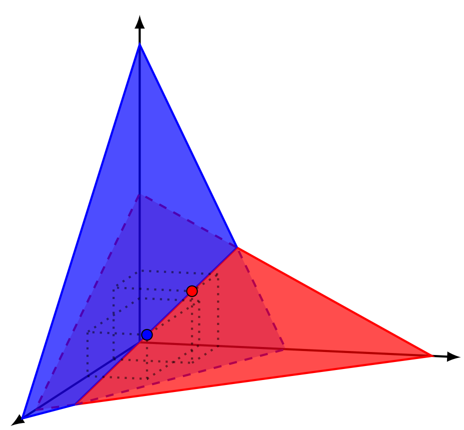
(a) Figure 1 in Chiappori and Rochet (1987). Non-invertible data set that is not smoothly rationalizable.



(b) Non-invertible data set that is smoothly rationalizable.



(c) Invertible data set (with indifferences) that is smoothly rationalizable



(d) Invertible data set (with indifferences) that is not smoothly rationalizable

Figure 1: Different combinations of invertibility and smooth rationalization of data sets

"flat" indifference set between them and, therefore, the same MRS. Panel (d) presents a data set that is both invertible and rationalizable but cannot be smoothly rationalized. As both choices are revealed indifferent to each other, each one has to be optimal from both budget sets; since achieving equality between one MRS and two different price ratios is impossible, no differentiable utility can rationalize the data.[8]

2 Data Set Modification

We start our study of smooth rationalization by proposing a modification of the data set. Such modification is the primary tool to develop our test for a differentiable utility.

Definition 7. *Let* $I(i) = \{j \in [N] : x^j \sim x^i\}$ *and*

$$r^i = \bigwedge_{j \in I(i)} p^j = \left(\min_{j \in I(i)} p_1^j, \ \min_{j \in I(i)} p_2^j, \ \ldots, \ \min_{j \in I(i)} p_K^j \right).$$

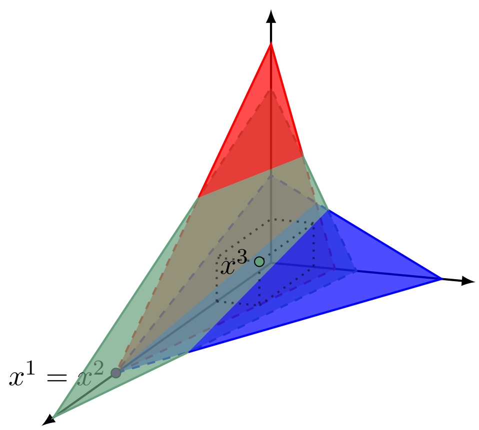
The modified data set *is* $\Gamma(\mathcal{D}) \triangleq (r^i, x^i)_{i \in [N]}$.

To construct the modified data set, we group all the choices that reveal indifferent to each other (note that the revealed indifference $\sim$ is an equivalence relation). Then, for each observation, replace its associated price with the meet among all the prices in its group. In contrast to the modified data set, we refer to $\mathcal{D}$ as the original data set.

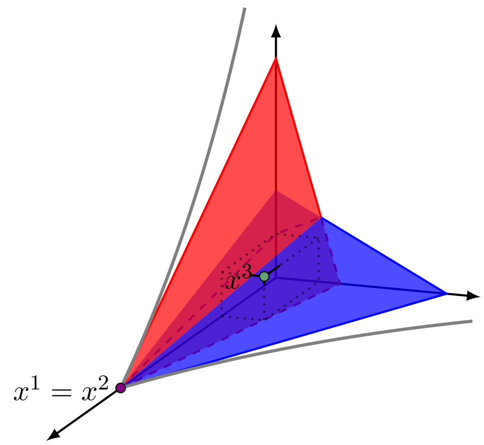
Figure 2 presents a simple data set that illustrates the role of the meet of prices among indifferent choices.[9] Panel (a) shows the original data set. As $x^1 = x^2$, these two choices revealed indifferent to each other, and since the data satisfies GARP, it is rationalizable. Furthermore, revealed preferences imply that any rationalizing utility U satisfies $U(x^3) > U(x^2) = U(x^1)$.

[8] In footnote 1, Chiappori and Rochet (1987) state that a data set that satisfies GARP and presents an invertible demand can be rationalized by a differentiable utility: *"(...) if the data satisfies both GARP and condition (ii) of Definition 3* $[p^i \neq p^j \implies x^i \neq x^j]$*, there exists an infinitely differentiable, strictly increasing, concave utility function which rationalizes them (...)"* Panel (d) of Figure 1 shows that such a statement is incorrect.

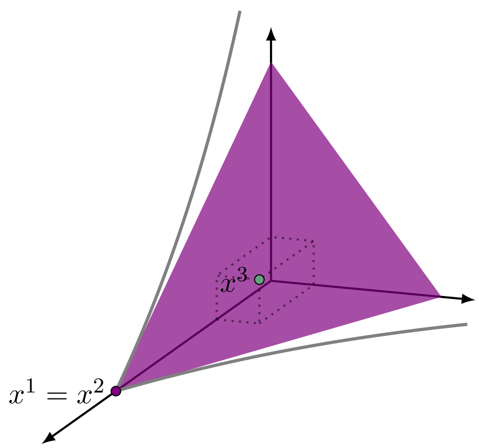
[9] The observations in this data set are $(p^1, x^1) = ((1/5, 4/3, 2/5), (5, 0, 0))$, $(p^2, x^2) = ((1/5, 1/2, 1), (1, 0, 0))$, and $(p^3, x^3) = ((1/7, 1, 1/2), (7/4, 1/2, 1/2))$.



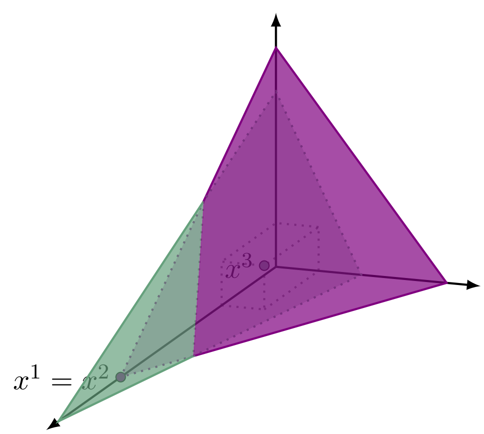
(a) $\mathcal{D}$ satisfies GARP. Revealed preferences imply $U(x^3) > U(x^1) = U(x^2)$.



(b) Intersection of x^1's indifference set with planes $x_2 = 0$ and $x_3 = 0$ (in gray).



(c) Plane below x^1's indifference set, which implies $U(x^1) = U(x^2) > U(x^3)$.



(d) $\Gamma(\mathcal{D})$ fails GARP

Figure 2: A rationalizable data set $\mathcal{D}$ for which $\Gamma(\mathcal{D})$ is not rationalizable.

In panel (b) of Figure 2, we focus on the indifference set of bundle $x^1(= x^2)$; specifically, we focus on the intersections of such indifference set with the $x_2 = 0$ and $x_3 = 0$ planes. The intersection between x^1's indifference set and the $x_2 = 0$ plane presents two properties: it is convex (on the plane) and does not intersect the interior of x^1's budget set. Similarly, as $x^1 = x^2$, the indifference set does not intersect the interior of x^2's budget set and is convex on the $x_3 = 0$ plane. The reason why x^1's budget set is the one binding along the $x_2 = 0$ plane is that, over that plane, the consumer can exchange x_1 for x_3, and x_3 is cheaper in the first observation (red budget set) than in the second (blue budget set), i.e., $p_3^1 < p_3^2$. Similarly, the blue budget set is the one bounding the intersection with the $x_3 = 0$ plane since, in this plane, the trade-off is between x_1 and x_2, and x_2 is cheaper in the second observation ($p_2^2 < p_2^1$). As $p_1^1 = p_1^2$, the slopes of the intersections at this bundle are given by p_3^1/p_1^1 (along the $x_2 = 0$ plane) and p_2^2/p_1^1 (along the $x_3 = 0$ plane). Formally, the bounds on the marginal utilities obtained by Γ are $\nabla U(x^i) \leq \lambda^i r^i$, where λ^i is the Lagrange multiplier associated with the budget constraint, i.e., the marginal utility of income (see Lemma 1 in Appendix B). These bounds motivate our focus on $\Gamma(\mathcal{D})$ to analyze smooth rationalization.

Suppose we try to rationalize the data in Figure 2 using a well-behaved differentiable utility U. Bounding the slopes of the indifference set, as we did in panel (b), is equivalent to bounding the marginal rates of substitution, i.e., the ratio of marginal utilities. Since U is differentiable and concave, we have $U(x^1) \geq U(x) + \nabla U(x^1) \cdot (x^1 - x)$ for every x. Combining this equation with the bounds on the marginal rate of substitutions (given by the minimum prices), we can bound the complete indifference set of x^1. Specifically, the indifference set has to be above the plane defined by the minimum prices; this is shown in panel (c) of the figure, which presents the price associated with the bundles x^1 and x^2 in the modified data set $\Gamma(\mathcal{D})$. The previous argument shows that every point within the (modified) budget set must have a lower utility than x^1. Finally, panel (d) shows the modified data set. As x^3 is in the interior of the (modified) budget set of x^1, we have $U(x^i) > U(x^3)$; similarly, revealed preferences imply $U(x^3) > U(x^1)$, a contradiction. As the modified data set fails GARP, we conclude that the original data set is not smoothly rationalizable.

The following result generalizes the relation between smooth rationalization and the data set modification Γ. Its proof is in Appendix B.

Proposition 1. *$\mathcal{D}$ is smoothly rationalizable by U if, and only if, $\Gamma(\mathcal{D})$ also is.*

Proposition 1 gives us a clear relation between smooth rationalization of the original data set and its modification: both conditions are equivalent. In particular, differentiability implies that the prices r^i in $\Gamma(\mathcal{D})$ generate tighter bounds in the vector of marginal utilities. The following result shows that such bounds are the only additional restrictions differentiability imposes. In other words, if the modification Γ does not modify the data set, then smooth rationalization imposes no restrictions on the observed data beyond the ones already imposed by rationalization. Furthermore, assuming the existence of higher-order derivatives in this case does not impose any restrictions.

Proposition 2. *Suppose $\mathcal{D} = \Gamma(\mathcal{D})$. Then $\mathcal{D}$ is rationalizable if, and only if, it is smoothly rationalizable by an infinitely differentiable utility.*

The proof of Proposition 2, presented in Appendix C, is constructive. Starting from a data set satisfying GARP and $\Gamma(\mathcal{D}) = \mathcal{D}$, we build a strictly increasing, concave, and infinitely differentiable U that rationalizes $\mathcal{D}$. We do so by combining a modified version of the Afriat inequalities (Lemma 3) with the convolution techniques proposed by Chiappori and Rochet (1987).

3 Smooth Rationalization

Our main result is a characterization of smooth rationalization through a simple revealed preference test. This characterization is an almost immediate consequence of Propositions 1 and 2. To test whether $\mathcal{D}$ is smoothly rationalizable, we first apply the transformation Γ to $\mathcal{D}$ until reaching a fixed point; this is, until reaching a data set $\mathcal{D}_\wedge$ satisfying $\Gamma(\mathcal{D}_\wedge) = \mathcal{D}_\wedge$.[10] By Proposition 2, smooth rationalization of $\mathcal{D}_\wedge$ is equivalent to $\mathcal{D}_\wedge$ satisfying GARP.

[10]Starting from any data set $\mathcal{D}$, a fixed point of Γ is assured to be reached in a finite number of steps by the following argument. In Definition 1, we construct the revealed preferences by looking at the cost of a bundle at a price and assuming an expenditure of 1. Although it is possible for a $\Gamma(\mathcal{D})$ to have $r^i \cdot x^i < 1$, we still construct this relation using an expenditure of one as a reference point; this is, x^i is revealed to x^j in $\Gamma(\mathcal{D})$ if $r^i \cdot x^j \leq 1$ even if $r^i \cdot x^i < 1$. The existence of the fixed point is assured since Γ expands the set of revealed indifferences. Therefore, in a worst-case scenario, the fixed point is reached when all choices are revealed indifferent to each other.

Furthermore, by Proposition 1 applied iteratively, smooth rationalization of $\mathcal{D}_\wedge$ and $\mathcal{D}$ is equivalent. Hence, $\mathcal{D}$ is smoothly rationalizable by U if, and only if, $\mathcal{D}_\wedge$ satisfies GARP. Furthermore, U can be chosen to be infinitely differentiable. This result is formalized in Theorem 1, where we also present a modified version of the Afriat inequalities.

Theorem 1. *Let $\mathcal{D}_\wedge$ be the fixed point of Γ, starting from $\mathcal{D}$. The following are equivalent:*

Sm-1) $\mathcal{D}$ is smoothly rationalizable.

Sm-2) $\mathcal{D}_\wedge$ is rationalizable (i.e., satisfies GARP).

Sm-3) Let $\sim_\wedge$ denote the revealed indifferences in $\mathcal{D}_\wedge$. There are numbers $u^i \in \mathbb{R}$, $\lambda^i > 0$ and K-dimensional vectors $\mu^i \geq \mathbf{0}$ such that

$$u^i > u^j + \lambda^i(1 - p^i \cdot x^j) + \mu^i \cdot x^j \qquad \text{whenever } x^i \not\sim_\wedge x^j \tag{G1}$$

$$u^i = u^j \qquad \text{whenever } x^i \sim_\wedge x^j \tag{G2}$$

$$\lambda^i p^i - \mu^i = \lambda^j p^j - \mu^j \qquad \text{whenever } x^i \sim_\wedge x^j \tag{G3}$$

$$\lambda^i p^i - \mu^i \gg \mathbf{0} \qquad \text{for all } i \in [N] \tag{G4}$$

$$\mu^i \cdot x^i = 0 \qquad \text{for all } i \in [N] \tag{G5}$$

Sm-4) $\mathcal{D}$ is smoothly rationalizable by an infinitely differentiable utility.

The proof of this result is in Appendix D. The equivalence between statements Sm-1) and Sm-2) gives us a simple test to check for the existence of a differentiable utility rationalizing the choices. After reaching the fixed point of Γ, the only further step is to check whether the resulting data set satisfies GARP.

The equivalence between Sm-1) and Sm-4) implies that whenever choices are smoothly rationalizable, such rationalization can be achieved by an infinitely differentiable utility. Hence, the existence of second- and higher-order derivatives of the utility function is not testable and can be assumed, from an empirical perspective, without loss of generality. From an applied perspective, this implies that if differentiability is assumed, tools like the Implicit Function Theorem or Taylor Expansions, which rely on higher-order derivatives, present no empirical restriction.

For an intuitive interpretation of Sm-3), we start from the classical consumer maximization problem. Take a consumer whose choices are driven by a strictly increasing, strictly concave, and differentiable utility U, subject to the budget constraint $p^i \cdot x \leq 1$, where x is a bundle of nonnegative commodities. Since U is concave and differentiable, we can solve this problem using the method of Lagrange multipliers and the Karush–Kuhn–Tucker conditions. The Lagrangian is

$$\mathcal{L}(x, \lambda, \mu) = U(x) + \lambda(1 - p^i \cdot x) + \mu \cdot x\,.$$

Although the function U is defined only on $\mathbb{R}^K_+$, to assure sufficiency of the first order conditions, we need to add Lagrange multipliers for the nonnegativity of x, as the equality between MRS and price ratio is not necessary for corner solutions. Let (x^i, λ^i, μ^i) be the optimal solution. The first order condition with respect to x is $\nabla U(x^i) = \lambda^i p^i - \mu^i$. Moreover, λ^i is the marginal utility of income, and each component of μ^i is the shadow cost of the corresponding nonnegativity constraint.

Starting from the consumer problem, set $u^i = U(x^i)$, and let λ^i and μ^i be the Lagrange multipliers. (G1) tells us that each choice is an optimal solution, as it maximizes the value of the Lagrangian. (G2) tells us that choices that reveal indifferent to each other must have the same utility level; furthermore, from (G3) we see that they also must have the same marginal utility (see the proof of Lemma 1 in Appendix B). (G4) is equivalent to the marginal utility being strictly positive, which follows from U being strictly increasing. Finally, (G5) is the complementary slackness condition from the non-negativity constraints.

The original Afriat inequalities are numbers $v^i \in \mathbb{R}$ and $\gamma^i > 0$ satisfying $v^i \geq v^j + \gamma^j(1 - p^i \cdot x^j)$. These numbers can also be interpreted as utility levels and marginal utilities of income. A relevant difference between the original Afriat inequalities and the ones in Sm-3) is that here we compare bundles using two relations, a strict inequality in (G1) and an equality in (G2). In contrast, the original rationalization result uses only one weak inequality. This difference arises because smooth rationalization can fail if we assume indifference between choices that are not revealed indifferent to each other, even if such an assumption does not contradict the revealed preferences. A simple case of this problem is presented in Figure 3, where adding the additional restriction of x^1 and x^2 being indifferent from each other rules out the possibility of smooth rationalization. Hence, the differentiability of the utility function imposes further restrictions on the ordering between choices than the ones in the revealed preferences. This is different from the case of simple rationalization, where we can achieve any ordering between the choices that do not violate the revealed preferences (Theorem 1 in Quah, 2014).

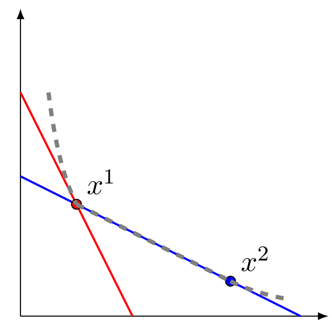


Figure 3: A rationalizing utility such that both choices are indifferent cannot be differentiable. Smooth rationalization requires $U(x^2) > U(x^1)$.

A final implication of Theorem 1 is that the only empirical component of a differentiable demand is the differentiability of the indifference sets. If we were to observe the agent's complete preference relation, smooth indifference sets are not sufficient for a smooth utility, as we also need the utility level to "vary smoothly" (Debreu, 1972). Dekel (1986) and Diasakos and Gerasimou (2022) present examples of preference relations whose indifference sets are smooth but cannot be represented by a differentiable utility. Furthermore, Neilson (1991) shows that differentiable indifference sets assure differentiability of the Hicksian demand but not of the Marshallian demand. On the other hand, a differentiable utility assures that both types of demand are differentiable. Such distinction cannot be made in empirical terms, as if the indifference sets are smooth, we can always construct a smooth utility.

Imposing More Structure on the Utility Function Theorem 1 characterizes smooth rationalization in the general case in which only minimal properties are assumed on the utility function. However, most applied economic research assumes further structure on the utility function. In the Online Appendix, we develop smooth rationalization tests for three families of widely used utility functions: strictly concave utilities, homothetic utilities, and quasilinear utilities. In all cases, show that the same approach can be used to characterize smooth rationalization. First, we focus on revealed indifferences to modify the data set. The main variation is that the indifferences we can infer depend on the specific structure we study; besides that, the modification always proceeds by taking the meet of the prices among revealed indifferences. In all cases, we also show that rationalization and smooth rationalization (by the specific family of utilities) are equivalent if the data set is invariant to the modification. Finally, we use these two results to develop characterizations of smooth rationalization (by the specific type of utilities) that are very similar to Theorem 1; in particular, a data set will be smoothly rationalizable by a utility of such family if, and only if, the fixed point of the modification is rationalizable by a utility of such family. Since we already know tests for these types of rationalization (from Matzkin and Richter, 1991, Varian, 1983, and Brown and Calsamiglia, 2007, respectively), the test is straightforward. In all cases, we show that the existence of higher-order derivatives has no empirical content and present a modified version of the Afriat inequalities as an alternative characterization.

4 Empirical Implementation

In this section, we study the empirical relevance of smooth rationalization. In particular, we study the differences between rationalization (i.e., GARP), smooth rationalization (Smooth GARP), and SSARP. Comparing rationalization and smooth rationalization tells us how restrictive it is (from an empirical perspective) to assume a differentiable utility. Comparing smooth rationalization and SSARP informs us about the relevance of Theorem 1 to test for a differentiable utility compared to the previous literature. Since our data set only includes choices between two goods and no subject chooses twice from the same price vector, GARP and SARP are equivalent. Hence, the difference between Smooth GARP and Strong SARP is only due to the latter generating false negatives.

We analyze the choices of 4,958 subjects from several experiments. They all follow the design of Choi et al. (2007b): subjects choose bundles composed of two different goods, and the choice is made graphically from a linear budget set. We analyze two different choice environments: risk and social. In the risk environment, there are two states of the world, and the goods are Arrow securities for each state. In the social environment, subjects play the *dictator game*, where the goods are the own consumption and consumption of an (anonymous) second subject. Subjects make multiple choices in each type of experiment from randomly drawn budget sets. After the experiment, one choice is randomly drawn from a uniform distribution, and payments are made according to that choice. For risk choices, one state of the world is also randomly drawn.

For each choice environment (Risk and Social), we split our data set into two samples: one in which subjects are either undergraduate or graduate students, which we denote Students, and another in which the subjects are representative of the general population, denoted General. The Risk-Students sample comprises 1,020 subjects, the Risk-General of 1,182, the Social-Students of 1,058, and the Social-General of 1,698. Subjects in the Risk-Students subjects make 25 choices each, and all others make 50 choices each. Appendix E presents a detailed description of the sources of each sub-sample.

4.1 Rationalizable Choices

We first focus exclusively on the subjects whose choices are rationalizable, which is the main topic of this paper. Table 2 presents summary statistics by data source. We observe that a vast majority (91.9%) of the rationalizable subjects satisfy Smooth GARP, i.e., have smoothly rationalizable choices. On the other hand, the percentage of these subjects who satisfy Strong SARP is significantly lower: 51.7%. Column (5) of the table measures the subjects who satisfy Smooth GARP and fail Strong SARP, as a percentage of the rationalizable subjects who fail Strong SARP. In other words, it measures, among the rationalizable subjects who fail Strong SARP, the percentage of subjects whose choices can be rationalized by a differentiable utility, i.e., the false negatives generated by Strong SARP. In total, 83% of rationalizable subjects who fail Strong SARP are false negatives, with wider differences in the social environment than in the risk one, and for students than for the general population. These result suggest that Strong SARP is inadequate to test for differentiability.

Regarding the difference between GARP and Smooth GARP, we find different patterns for the social and risk environments. In the social environment, the the general sample is

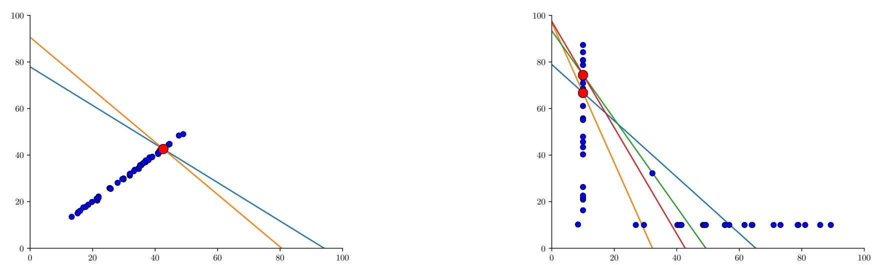
Table 2: Subjects Satisfying Axioms by Data Source

	(1)	(2)	(3)	(4)	(5)
Source	N	$\frac{\text{GARP}}{\text{N}}$	$\frac{\text{Smooth}}{\text{GARP}}$	$\frac{\text{SSARP}}{\text{GARP}}$	$\frac{\text{Smooth - SSARP}}{\text{GARP - SSARP}}$
Risk - Students	1020	25.3%	91.1%	65.5%	74.2%
Risk - General	1182	19.5%	91.8%	88.3%	29.7%
Risk - All	2202	22.2%	91.4%	76.3%	63.8%
Social - Students	1058	32.7%	96.8%	25.1%	95.8%
Social - General	1698	8.6%	82.2%	32.2%	73.7%
Social - All	2756	17.9%	92.5%	27.2%	89.7%
TOTAL	4958	19.8%	91.9%	51.7%	83.3%

*N is the number of subjects; GARP is the number of rationalizable subjects; Smooth is the number of smoothly rationalizable subjects; SSARP is the number of subjects who satisfy this axiom. Last column measures share of smoothly rationalizable subjects among rationalizable subjects who fail SSARP.

less likely to satisfy Smooth GARP due to a more prevalent presence of choices resembling a Leontief utility, which is interpreted as subjects having egalitarian preferences.[11]

For the case of risk choices, we find two patterns of choice consistent with classical models of choice under risk among rationalizable subjects who fail smooth GARP. The first type Leontief choices, which are interpreted in this environment as having infinite risk aversion. The second type is subjects who present safety-first preferences *a la* Roy (1952); these subjects choose by first assuring a certain level of consumption in both states and beyond that level, act as almost-risk neutral. Figure 4 presents examples of subjects who resemble these two utility functions.[12]



(a) Subject 109 from Choi et al. (2007a)

(b) Subject 39 from Cappelen et al. (2021)

Figure 4: Examples of choices that resemble common non-differentiable utilities. The subject in (a) mimics Leontieff preferences, and the one in (d) safety-first preferences (Roy, 1952).

[11] Since choices are made graphically, we measure Leontieff choices with slack: we classify a choice as Leontieff if the ratio x_1/x_2 is above $1/1.05$ and below 1.05. For both Risk and Social environments, the General sample presents a higher share of Leontief choices than the Students one, but this difference is higher in the Social environment than in the Risk one. For the Risk environment, the General sample has 39.7% of Leontieff choices, and the Students sample has 21.1%. In the Social environment, such choices are 30.3% in the General sample and 3.0% in the Students sample.

[12] Additionally, a minority of subjects present combinations of risk neutrality (i.e., corner solutions) for extreme price ratios and infinite risk aversion for price ratios closer to one. Choi et al. (2007a) describe these subjects as presenting loss or disappointment aversion, with the 45-degree line being the reference point.

4.2 Nonrationalizable Choices

We recover preferences of subjects who fail GARP using the Houtman and Maks (1985) Index (HM Index). The HM Index looks for the smallest subset of observations that needs to be removed such that the remaining observations satisfy the axiom in question (in our case, either GARP, Smooth GARP, or Strong SARP).[13] Usually, the HM Index, as other measures of distance from GARP, is interpreted as a measure of economic rationality (see Kariv and Silverman, 2013, for a discussion of this topic).

The choice of the HM Index is for practical purposes. The most popular methods to recover non-rationalizable preferences are motivated by the idea of *partial efficiency* (Afriat, 1973).[14] Along such methods, the HM Index is the only one that allows to differentiate between GARP, Smooth GARP, and Strong SARP (see Ugarte, 2023b). To compute the HM Index, we rely on the techniques introduced by Demuynck and Rehbeck (2023).

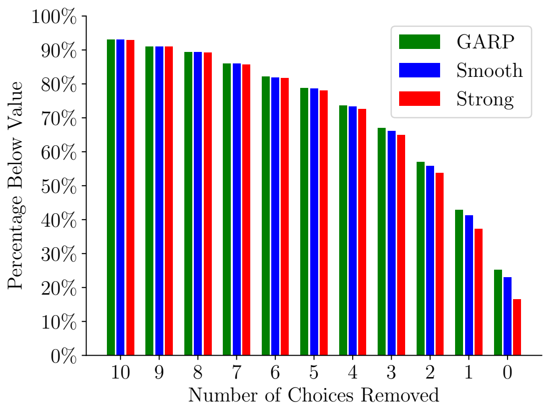
Figure 5 presents the (reversed) cumulative distribution of the HM Index, differentiated by sub-sample. Since the HM Index measures distance from economic rationality, subjects with a lower index value (to the right of the graph) are closer to economic rationality (i.e., closer to satisfying GARP). Each bar represents the subjects with an HM Index that is less or equal to the given value, i.e., subjects at that distance or closer to rationalization. In particular, the bars at the zero level of the index represent the subjects consistent with the corresponding axiom, hence coincide with the numbers in Table 2.

The HM Index for GARP and Smooth GARP behave similarly for all samples. This result suggests that the cost of adding differentiability to the utility function is small. Furthermore, this difference decreases as we increase the cutoff value of the HM Index. One possible explanation is that subjects further away from GARP (i.e., with a higher HM Index) may present more erratic behavior, making it more difficult to reject Smooth GARP and Strong SARP.

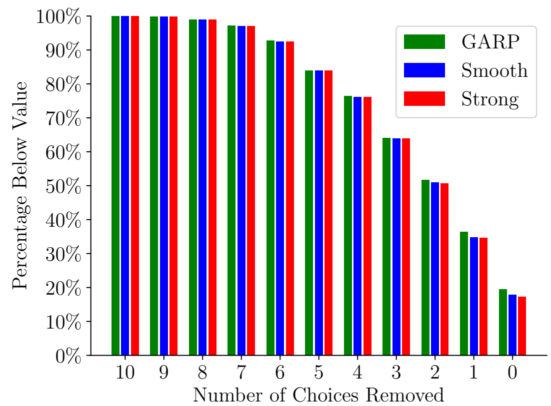
The main differences in Figure 5 are between choice environments. The Social envi-

[13] Given a data set $\mathcal{D}$ and an axiom $\mathcal{A}$, $HM(\mathcal{D},\mathcal{A}) = \min_{E\subset[N]} |E|$ such that $(p^i, x^i)_{i\in[N]\setminus E}$ satisfies $\mathcal{A}$.

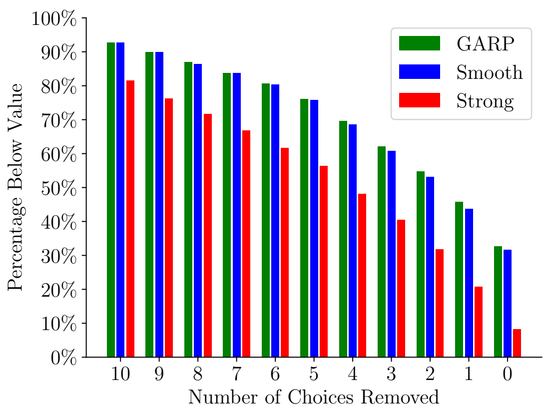
[14] Besides the HM Index, the two most popular methods to measure distance from GARP, both based in partial efficiency are the Critical Cost Efficiency Index (Afriat, 1973) and the Varian (1990) Index. de Clippel and Rozen (2021) and Ugarte (2023a) propose methods to recover preferences that do not rely on partial efficiency.



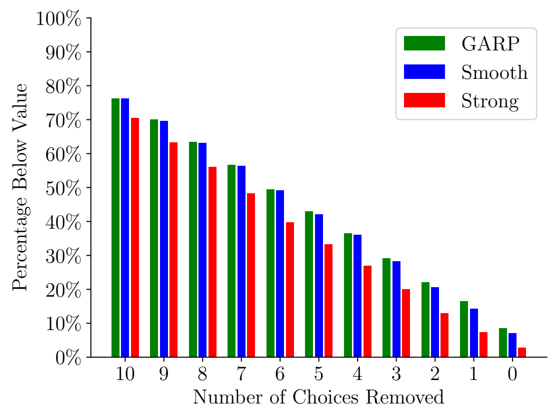
(a) Risk - Students (50 choices)



(b) Risk - General (25 choices)



(c) Social - Students (50 choices)



(d) Social - General (50 choices)

Figure 5: Percentage of Subjects with Houtman and Maks (1985) Index below value for each sub sample and different axioms.

ronment generally presents a higher distance between Smooth GARP and Strong SARP. Furthermore, although it decreases, such distance is still considerable when we include subjects with higher HM Index. On the one hand, the share of subjects with HM Index below a cutoff in the Risk environment tends to equalize for both axioms as we increase the value of such cutoff; on the other, such a difference remains significant for the social environment. For students, the HM Indices for GARP and Smooth GARP behave similarly (in terms of their cumulative distribution), and the main difference is in the value of the HM Index for Strong SARP. For the general population, the index is lower (i.e., a higher share of subjects are below a given value) in the Risk than in the Social environment.[15]

Our empirical results present two consistent patterns. The first one is that the differences between Smooth GARP and SARP seem small but not zero. Hence, the assumption of differentiability is not general empirically but encloses a high share of the subjects. Second, our results suggest that Strong SARP is too restrictive to test for a differentiable utility: many subjects that satisfy Smooth GARP fail Strong SARP.

5 Concluding Remarks

Applied models in economics often assume that consumers have a differentiable utility function, which adds tractability and simplifies comparative statics. However, the behavioral restrictions of this assumption remain elusive. In this paper, we characterize the requirements that differentiability imposes on the behavior of a competitive consumer endowed with a well-behaved (continuous, increasing, and concave) utility.

If a data set is rationalizable, the only additional requirement to rationalize it with a differentiable utility is for indifferences curves to be smooth. Using revealed indifferences, we propose a simple modification of the data set, which tightens the bounds on the marginal rates of substitution. Moreover, we show that such tightening is the only restriction differentiability imposes on finite data. Our main theorem uses these results to characterize rationalization by a well-behaved and differentiable utility (i.e., *smooth rationalization*) in a form that can be easily tested. We also present an alternative characterization through

[15]Since in the Risk-General sample subjects make only 25 choices each, removing one observation is, as a share, twice as relevant as in the other samples.

a modified version of the classical Afriat inequalities. Finally, we show that the existence of second- and higher-order derivatives is, from an empirical standpoint, an assumption that can be made without loss of generality. Higher-order derivatives are a widespread assumption as they facilitate comparative statics.

We test smooth rationalization on experimental data. Our results suggest that assuming a differentiable utility is not particularly restrictive, as most rationalizable subjects are also smoothly rationalizable. This similarity holds when the rationalization requirements are relaxed. Finally, we find substantial differences with the sufficient conditions developed by Chiappori and Rochet (1987), which shows the empirical relevance of our result.

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A APPENDIX

B Proof of Proposition 1

The proof of Proposition 1 relies on the first order conditions of the utility maximization problem. Take the problem of maximizing a concave and differentiable utility $U(x)$, subject to the budget constraint $p^i \cdot x \leq 1$ and the non-negativity constraints $x \geq \mathbf{0}$. We can solve this problem using the method of Lagrange multipliers and the Karush–Kuhn–Tucker conditions. The Lagrangian is $\mathcal{L}(x, \lambda, \mu) = U(x) + \lambda(1 - p \cdot x) + \mu \cdot x$, where $\lambda > 0$ is the multiplier associated with the budget set, and $\mu \in \mathbb{R}^K_+$ is the vector of multipliers associated with the non-negativity conditions. We denote the optimal values associated with this problem by x^i, λ^i, μ^i. These values satisfy the first order condition associated with x, $\nabla U(x^i) - (\lambda^i p^i - \mu^i) = 0$, the complementary slackness condition of the non-negativity restrictions $\mu^i \cdot x^i = 0$, and, as U is strictly increasing, $\lambda^i > 0$. Furthermore, the first two equations imply $x^i \cdot \nabla U(x^i) = \lambda^i$.

Lemma 1. *Suppose $\mathcal{D}$ is smoothly rationalized by U, and let λ^i be the marginal utility of money (i.e., the Lagrange multiplier of the budget set) when choosing x^i from price p^i. Let $\Gamma(\mathcal{D}) = (r^i, x^i)$. Then $\nabla U(x^i) \leq \lambda^i r^i$.*

Proof. Suppose $\mathcal{D}$ is smoothly rationalizable by U, and let λ^i and μ^i be the Lagrange multipliers associated with observation i. Take $i, j \in [N]$ such that $x^i \sim x^j$. As $x^i \succsim x^j$ and $x^j \succsim x^i$ there are sequences of observations $(m_\ell)_{\ell \in [L]}$ and $(n_s)_{s \in [S]}$ such that

$$x^i \succsim^\star x^{m_1} \succsim^\star \ldots \succsim^\star x^{m_L} \succsim^\star x^j \succsim^\star x^{n_1} \succsim^\star \ldots \succsim^\star x^{n_S} \succsim^\star x^i \,. \tag{1}$$

Since $\mathcal{D}$ satisfies GARP, revealed preferences imply $U(x^i) = U(x^j) = U(x^{m_\ell}) = U(x^{n_s})$ for all $\ell \in [L]$, $s \in [S]$.

Since $x^i \succsim^\star x^{m_1}$, we have $p^i \cdot x^{m_1} \leq 1$, and since $x^{m_1} \succsim x^i$ and $\mathcal{D}$ satisfies GARP, $x^i \not\succ^\star x^{m_1}$; therefore $p^i \cdot x^{m_1} = 1$. As $U(x^i) = U(x^{m_1})$, Theorem 2.2 in Jeyakumar et al. (2004) implies $\nabla U(x^i) = \nabla U(x^{m_1})$. Therefore

$$\lambda^i p^i + \mu^i = \nabla U(x^i) = \nabla U(x^{m_1}) = \lambda^{m_1} p^{m_1} - \mu^{m_1} \,. \tag{2}$$

Theorem 2.2 in Jeyakumar et al. (2004) also implies $\mu^i \cdot x^{m_1} = 0$. Since $p^i \cdot x^{m_1} = 1$, the dot product of (2) and x^{m_1} yields $\lambda^i = \lambda^{m_1}$.

Applying the previous argument iteratively to (1) we obtain $\nabla U(x^i) = \nabla U(x^{m_1}) = \ldots = \nabla U(x^{m_L}) = \nabla U(x^j)$ and $\lambda^i = \lambda^{m_1} = \ldots = \lambda^{m_L} = \lambda^j$. Hence $\nabla U(x^i) = \nabla U(x^j) = \lambda^j p^j - \mu^j \leq \lambda^j p^j = \lambda^i p^j$ whenever $x^i \sim x^j$. As j is arbitrary and U is strictly increasing, the definition of r^i implies the desired result. □ □

Proof of PROPOSITION 1. For necessity suppose $\mathcal{D}$ is smoothly rationalizable by U. As U is concave

$$U(x^i) - U(x) \geq \nabla U(x^i) \cdot (x^i - x) = \lambda^i - \nabla U(x^i) \cdot x \geq \lambda^i(1 - r^i \cdot x) \tag{3}$$

The second equality follows from $x^i \cdot \nabla U(x^i) = \lambda^i$, and the third from Lemma 1 and $x \geq \mathbf{0}$. As $\lambda^i > 0$, (3) implies that $U(x^i) \geq U(x)$ whenever $r \cdot x \leq 1$. Therefore U rationalizes $\Gamma(\mathcal{D})$.

For sufficiency suppose that U rationalizes $\Gamma(\mathcal{D})$. For every $i \in [N]$ let $\hat{\lambda}^i > 0$ and $\hat{\mu}^i \geq \mathbf{0}$ be the Lagrange multipliers in the utility maximization problems of $\Gamma(\mathcal{D})$. Let $\mu^i = \hat{\lambda}^i \cdot p^i - \nabla U(x^i) \geq \hat{\mu}^i \geq \mathbf{0}$. It is easy to check that $(x^i, \hat{\lambda}^i, \mu^i)_{i\in[N]}$ satisfy all the Karush-Kuhn-Tucker conditions of the maximization of U in $\mathcal{D}$, which are sufficient as U is concave. Therefore U rationalizes $\mathcal{D}$. □ □

C Proof of Proposition 2

Lemma 2. *For some $i \in [N]$, $p^j \cdot x^m > 1$ whenever $x^j \sim x^i$ and $x^m \not\sim x^i$.*

Proof. By contrapositive, suppose for every $i \in [N]$ there are $x^j \sim x^i$ and $x^m \not\sim x^i$ such that $p^j \cdot x^m \leq 1$. Then $x^i \succsim x^j \succsim^\star x^m$, hence $x^i \succsim x^m$. Thus, $x^m \not\sim x^i$ implies $x^m \not\succsim x^i$. Hence, we can construct an infinite sequence $(n_\ell)_{\ell=1}^\infty$ such that, for every ℓ, $x^{n_\ell} \succsim x^{n_{\ell+1}}$ and $x^{n_{\ell+1}} \not\succsim x^{n_\ell}$. As $\mathcal{D}$ is finite, there is an observation that repeats in the sequence, i.e., there are $r, s \in \mathbb{N}$ such that $r + 1 < s$, and $x^{n_r} = x^{n_s}$. Then $x^{n_{r+1}} \succsim x^{n_s} = x^{n_r}$, a contradiction. □ □

Lemma 3. *If GARP holds there are numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ such that $x^i \not\sim x^j$ implies $u^i > u^j + \lambda^i(1 - p^i \cdot x^j)$, and $x^i \sim x^j$ implies $u^i = u^j$ and $\lambda^i = \lambda^j$.*

Proof. We proceed by induction on N. If $N = 1$, set $u^1 = \lambda^1 = 1$.

Suppose GARP holds for all databases comprised of $N-1$ or less observations, and take $\mathcal{D}$ comprised of N observations. By Lemma 2, and without loss of generality, suppose N is such that $p^i \cdot x^j > 1$ whenever $x^i \sim x^N$ and $x^j \not\sim x^N$. If $x^i \sim x^N$ for all i, set $u^i = \lambda^i = 1$ for every $i \in [N]$. Then the conditions hold.

If there is j such that $x^j \not\sim x^N$, then the data set $(p^j, x^j)_{\{j:x^j \not\sim x^N\}}$ is a data set of $N-1$ or less observations for which the conditions hold. Take $\varepsilon > 0$, and for every i such that $x^i \sim x^N$ set

$$u^i = \min_{\{m:x^m \sim x^N\}} \min_{\{j:x^j \not\sim x^N\}} u^j - \lambda^j(1 - p^j \cdot x^m) - \varepsilon\,.$$

As $\sim$ is an equivalence relation, $u^i = u^j$ whenever $x^i \sim x^j$. Moreover, whenever $x^j \not\sim x^N$ and $x^m \sim x^N$ we have $u^m \leq u^j - \lambda^j(1 - p^j \cdot x^m) - \varepsilon < u^j - \lambda^j(1 - p^j \cdot x^m)$. Whenever $x^i \sim x^N$ set

$$\lambda^i = \max\left\{\max_{\{m:x^m \sim x^N\}} \max_{\{j:x^j \not\sim x^N\}} \frac{u^j - u^m}{p^m \cdot x^j - 1} + \varepsilon;\, 1\right\}\,.$$

Hence $\lambda^i = \lambda^N > 0$ whenever $x^i \sim x^N$. Finally, if $x^i \sim x^N$ and $x^j \not\sim x^N$ it follows form the definition of λ^i and $p^i \cdot x^j > 1$ that $u^j + \lambda^i(1 - p^i \cdot x^j) < u^i$. □ □

Proof of PROPOSITION 2. Sufficiency is immediate. For necessity suppose $\mathcal{D}$ satisfies GARP. Take the numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ from Lemma 3 and for each i define the function $\phi : \mathbb{R}^K \to \mathbb{R}$ by $\phi^i(x) = u^i - \lambda^i(1 - p^i \cdot x)$, which is continuous, concave, and strictly increasing. Moreover, $\mathcal{D} = \Gamma(\mathcal{D})$ implies $p^i = p^j$ whenever $x^i \sim x^j$, hence $\phi^i = \phi^j$ whenever $x^i \sim x^j$. Let $V(x) = \min_{i \in [N]} \phi^i(x)$, which is also continuous, strictly increasing, and concave.

By Lemma 3 we have $V(x^i) = \phi^i(x^i)$ for all i and $V(x^i) < \phi^m(x^i)$ whenever $x^m \not\sim x^i$. Denote by $B(\zeta)$ the open ball of radius ζ centered at $\mathbf{0}$. Continuity of V implies that there is $\eta > 0$ such that $V(x^i - x) = \phi^i(x^i - x)$ for all $i \in [N]$ and $x \in B(\eta)$. Define the functions

$$\rho(x) = \mathbb{I}_{\{||x||<1\}}\left[\int_{B(1)} \exp\left(-\frac{1}{||y^2||-1}\right) dy\right]^{-1} \exp\left(-\frac{1}{||x^2||-1}\right);$$

$$\rho_\eta(x) = \frac{1}{\eta}\rho\left(\frac{x}{\eta}\right); \text{ and}$$

$$\tilde{U}(x) = (V \star \rho_\eta)(x) = \int_{\mathbb{R}^K_+} V(x - \xi)\rho_\eta(\xi)d\xi = \int_{B(\eta)} V(x - \xi)\rho_\eta(\xi)d\xi\,.$$

Then $\tilde{U} : \mathbb{R}^K \to \mathbb{R}$ is strictly increasing, concave, and infinitely differentiable. Moreover, $\int_{B(\eta)} \rho_\eta(\xi) d\xi = 1$, and $\int_{B(\eta)} \xi \rho_\eta(\xi) d\xi = \mathbf{0}$ (Chiappori and Rochet, 1987). We have

$$\begin{aligned}
\tilde{U}(x^i) &= \int_{B(\eta)} \phi^i(x^i - \xi)\rho_\eta(\xi)\, d\xi \\
&= \left[u^i - \lambda^i(1 - p^i \cdot x^i)\right] \int_{B(\eta)} \rho_\eta(\xi)\, d\xi - \lambda^i p^i \cdot \int_{B(\eta)} \xi \rho_\eta(\xi)\, d\xi \\
&= u^i
\end{aligned} \tag{4}$$

The first line follows from $V(x^i - x) = \phi^i(x^i - x)$ whenever $x \in B(\eta)$; the second one replaces the definition of ϕ^i and splits terms; and the last one replaces $p^i \cdot x^i = 1$, $\int_{B(\eta)} \rho_\eta(\xi) d\xi = 1$, and $\int_{B(\eta)} \xi \rho_\eta(\xi) d\xi = \mathbf{0}$. Take x such that $p^i \cdot x \leq 1$. Then

$$\begin{aligned}
\tilde{U}(x) &= \int_{B(\eta)} V(x - \xi)\rho_\eta(\xi)\, d\xi \\
&\leq \left[u^i - \lambda^i(1 - p^i \cdot x)\right] \int_{B(\eta)} \rho_\eta(\xi)\, d\xi - \lambda^i p^i \cdot \int_{B(\eta)} \xi \rho_\eta(\xi)\, d\xi \\
&\leq u^i\,.
\end{aligned}$$

where second line follows from $V(x) \leq \phi^i(x)$, replaces the definition of ϕ^i, and splits terms, and the last one follows from $p^i \cdot x \leq 1$, $\int_{B(\eta)} \rho_\eta(\xi) d\xi = 1$, and $\int_{B(\eta)} \xi \rho_\eta(\xi) d\xi = \mathbf{0}$. The last equation and (4) imply $\tilde{U}(x^i) \geq \tilde{U}(x)$ whenever $p^i \cdot x \leq 1$. Finally, let $U : \mathbb{R}^K_+ \to \mathbb{R}$ be the restriction of $\tilde{U}$ to $\mathbb{R}^K_+$. U is infinitely differentiable and smoothly rationalizes $\mathcal{D}$. □ □

D Proof of Theorem 1

That Sm-1) implies Sm-2) follows from a iterative application of Proposition 1. To see that Sm-2) implies Sm-3), let q^i be the prices in $\mathcal{D}_\wedge$, take the numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ from Lemma 3 applied to $\mathcal{D}_\wedge$, and define $\mu^i = \lambda^i(p^i - q^i)$, where $(q^i)_{i\in[N]}$ are the prices in $\mathcal{D}_\wedge$. It is easy to check that all the conditions are satisfied. Starting from Sm-3), a construction similar to the one in the proof of Proposition 2, with the only difference that in this case $\phi^i(x) = u^i - \lambda^i(1 - p^i \cdot x) - \mu^i \cdot x$, yields an infinitely differentiable utility that smoothly rationalizes $\mathcal{D}$. Finally, the proof that Sm-4) implies Sm-1) is immediate. □

E Sample Description

The four sub-samples are:

- Risk-Students: 1,020 subjects making 50 choices each. Of the total, 974 subjects make choices in a symmetric environment, where both states of the world are equally likely. The remaining 46 subjects face an asymmetric environment where the probabilities are $2/3$ and $1/3$. The symmetric data is taken from Choi et al. (2007a); Zame et al. (2020); Cappelen et al. (2021); Dembo et al. (2021), and the asymmetric one from Choi et al. (2007a). All the subjects are undergraduate students in different universities.

- Risk-General: 1,182 subjects making 25 choices each. The source of this data set is the experiment in Choi et al. (2014), where all subjects face a symmetric environment. The sample is representative of the Dutch-speaking population in the Netherlands.

- Social-Students: 1,058 subjects making 50 choices each. The data is taken from Fisman et al. (2007, 2015b,a); Li et al. (2017).

- Social-General: this sample comprises 1,698 subjects making 50 choices each. The data is taken from Fisman et al. (2017, 2022). Both experiments are embedded in the American Life Panel (ALP), an internet survey administered by the RAND Corporation to adult Americans.