The generality of the Strong Axiom

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May 5, 2023

Abstract

Economic research usually endows consumers with a single-valued demand function. When choices are rationalizable, this assumption can be tested by the Strong Axiom of Revealed Preferences, SARP, as if they fail such a test, the demand is set-valued. We extend this test to non-rationalizable choices using partial efficiency, the most popular method to recover preferences. Under partial efficiency, a single-valued demand cannot be tested; furthermore, it can always be chosen to be infinitely differentiable. Hence, the existence of a single-valued, infinitely differentiable demand is falsified if, and only if, choices are rationalizable but fail SARP, which we do not observe in laboratory data. From an empirical standpoint, our results suggest that assuming a differentiable demand does not carry a cost.

EconLit Codes: D01 (Microeconomic Behavior: Underlying Principles), D12 (Consumer Economics: Empirical Analysis), D90 (Micro-Based Behavioral Economics: General), C91 (Design of Experiments: Laboratory, Individual)

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1 Introduction

One of the most widespread assumptions in economic research is to endow agents with (single-valued) demand functions instead of (set-valued) correspondences. This assumption simplifies the analysis in both theoretical and empirical research. Most results in general equilibrium, applied game theory, and mechanism and information design rely upon this assumption to keep the models tractable. Furthermore, demand functions are usually assumed to present some level of smoothness to simplify comparative statics. Empirically, demand estimations typically proceed by adding an error term to a parametric demand satisfying such characteristics.

In this paper, I study the empirical content of both assuming a single-valued demand and its differentiability. I extend the classical analysis to the case when the agent's choices are an imperfect implementation of her preferences and therefore fail the Generalized Axiom of Revealed Preferences (GARP). In other words, I study the possibility of using an agent's (possibly inconsistent) observed choices to falsify the properties of her demand. The main result of this paper shows that if the agent fails GARP, then it is impossible to falsify the demand being a function. Furthermore, such a function can always be chosen to be infinitely differentiable.

From Afriat (1967) and Varian (1982), we know that a consumer's choice data can be thought of as being a perfect implementation of a utility function if and only if it satisfies GARP. However, the demand derived from the utility function recovered with Afriat's method is neither single-valued (i.e., it is not a function) nor differentiable. Matzkin and Richter (1991) show that a demand function (instead of a correspondence) exists if, and only if, choices satisfy Houthakker's (1950) Strong Axiom of Revealed Preferences (SARP). Moreover, Lee and Wong (2005) show that under SARP, we can always choose such demand to be infinitely differentiable.

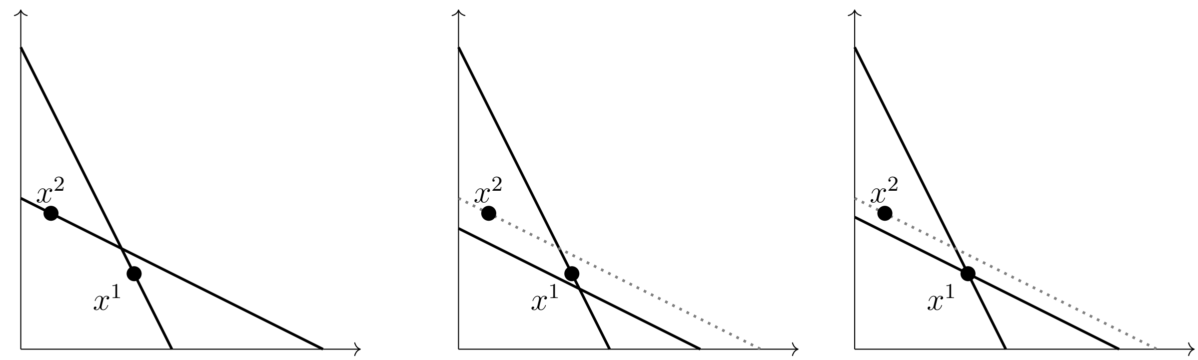
I extend the analysis in Matzkin and Richter (1991) and Lee and Wong (2005) to non-rationalizable choices, i.e., choices that fail (GARP). Our interpretation of non-rationalizable choices is that the agent has an underlying preference but presents some form of bounded rationality *a la* Simon (1955). We focus on recovering preferences using partial efficiency

(Afriat, 1973; Varian, 1990; Halevy et al., 2018), the most popular method to analyze choices that fail GARP non-parametrically. The main results show that whenever choices fail GARP, neither the assumption of a demand function nor the differentiability of such function can be falsified.

When a consumer's choices are suboptimal, GARP (and SARP) are insufficient to learn about her preferences. Halevy et al. (2018) propose a method to recover preferences under bounded rationality. First, they show a modified version of the Afriat Theorem that integrates partial efficiency. Intuitively, partial efficiency requires a choice to be preferred not to every feasible alternative but only to those whose cost is a share of the consumer's income. Formally, take a data set of N observations, where each observation i is a price vector p^i and a choice x^i; partial efficiency $v_i \in [0,1]$ in choice i requires x^i to be preferred only to bundles whose cost is $v_i\, p^i\, x^i$ instead of $p^i\, x^i$ (if $v_i = 1$ for all the observations, then we go back to the classical, i.e., full efficiency, definition of GARP). Using this idea, they propose to recover preferences by, according to a cost function, choosing the partial-efficiency levels that satisfy (a partial-efficiency version of) GARP at a minimum cost.

This paper extends the work in Halevy et al. (2018) to analyze the empirical content of a demand function in this setting. First we show that the equivalence between SARP and a demand function does not hold under partial efficiency. Specifically, the partial-efficiency version of SARP is a sufficient but not necessary condition to rationalize the data with a utility generating a demand function. Furthermore, the same holds if the utility is required to generate an infinitely differentiable demand.

Our results show that if the data fails GARP, then, under partial efficiency, the existence of an infinitely differentiable demand cannot be falsified. Specifically, suppose the data fails GARP, then for any utility that rationalizes the data under partial efficiency, there is another one that (1) also rationalizes the choices, (2) generates a single-valued and infinitely differentiable demand, and (3) yields the same partial-efficiency loss. Figure 1 presents an intuitive explanation of this result. In (a), we see choice data that fails GARP: x^1 is (revealed) strictly preferred to x^2, and x^2 is (revealed) strictly preferred to x^1. To rationalize the data, we need to add partial efficiency to one choice, and we do it to x^2 since it requires a smaller shrink of the budget set (the cost of x^1 when x^2 is chosen is a higher share of the income than the cost of x^2 when x^1 is chosen). In (b), we shrink the budget set of x^2 such that x^1 is outside this new budget set, then the data satisfies both GARP and SARP, as x^1 is revealed preferred to x^2 and x^2 is not revealed preferred to x^1. However, whenever x^1 is in the (modified) budget set of x^2, even if it is in the upper boundary as in (c), the data will fail both GARP and SARP.



(a) Data fails GARP and SARP

(b) Data satisfies (partial efficiency) GARP and SARP

(c) Data fails (partial efficiency) GARP and SARP

Figure 1: An intuitive explanation of the main result. In (a) GARP does not hold, so we need partial efficiency to rationalize the data. In (b) x^1 is not in the (relaxed) budget set of x^2 and both GARP and SARP hold. If x^1 is in the upper boundary of x^2, as in (c), both GARP and SARP fail.

From Afriat (1967), and Matzkin and Richter (1991), we know that if the data satisfies GARP, the existence of a single-valued demand can be falsified through SARP. Our results complete this test by adding that whenever the data fails GARP, it is always possible to recover a utility function generating a demand function. Furthermore, as in the case of (partial-efficiency) SARP, we show that such utility can always be chosen to generate an infinitely differentiable demand. Having a complete test, we empirically analyze its existence using experimental data from 322 individuals (50 choices each). For none of them, we can rule out a utility generating an infinitely differentiable demand function. Such a result suggests that this widespread assumption does not carry a cost from an empirical standpoint.

1.1 Related Literature

The idea of revealed preferences traces back to Samuelson (1938). Afriat's (1967) seminal paper shows that observed choices can be thought of as generated by a continuous, strictly increasing, and concave utility if, and only if, they satisfy an easy-to-check condition called cyclical consistency. The most famous version of this condition is GARP, proposed by Varian (1982). Matzkin and Richter (1991) show that SARP, a test proposed by Houthakker (1950), is equivalent to a strict concave utility, therefore generating a demand function. Lee and Wong (2005) strengthen Matzkin and Richter's (1991) result by showing that the same test is sufficient for the utility to generate an infinitely differentiable demand. Revealed preferences analysis has been extended in several directions: Chiappori and Rochet (1987) and Ugarte (2023) study the differentiability of the utility function, Forges and Minelli (2009) study non-linear budget sets, Reny (2015) studies infinite datasets, and Nishimura et al. (2017) study general choice environments and different criteria for objectively better bundles.

The literature studying non-rationalizable choices, i.e., choices that fail GARP, starts with Afriat (1973). He proposes to use the same level of partial efficiency in all observations to measure the distance from economic rationality. After him, several other measures have been proposed noticing that different decisions can use different partial efficiency levels (Houtman & Maks, 1985; Varian, 1990; Echenique et al., 2011; Dean & Martin, 2016). Polisson et al. (2020) uses the same idea to study distance from expected-utility models. Methods that do not rely on partial efficiency have been proposed only recently (de Clippel & Rozen, 2021; Echenique et al., 2022; Ugarte, 2022), to the point that, to the best of our knowledge, there are no empirical papers in the revealed preference literature that does not rely on these methods. Halevy et al. (2018) take a further step and investigate how to use partial efficiency to recover preferences, focusing specifically on the Varian (1990) Index. The analysis in Halevy et al. (2018) is the starting point of this paper.

The rest of the paper proceeds as follows. Section 2 presents the problem and analyzes conditions to recover preferences generating an infinitely differentiable demand, given a partial efficiency level. Section 3 shows how to use the Varian Index to choose the level of partial efficiency, characterizes the test for the existence of a differentiable demand function under partial efficiency, and implements this test in laboratory data. Finally, Section 4 concludes. All proofs are in the Appendix.

2 Data Rationalization under Partial Efficiency

2.1 Setup

Consider an agent who consumes bundles of K commodities and makes N choices.[1] In each choice $i \in [N]$, she faces a price vector $p^i \in \mathbb{R}^K_{++}$ and chooses a bundle x^i from the budget set $\left\{x \in \mathbb{R}^K_+ : p^i x \leq 1\right\}$ (the normalization of income to 1 is without loss of generality). Together, prices and bundles form the data set $\mathcal{D} = (p^i, x^i)_{i \in [N]}$, which is the primitive of our problem. We refer to the bundles in $\mathcal{D}$ as choices. As standard in the revealed preference literature, we assume that the agent spends all her income, i.e., $p^i\, x^i = 1$.

From Afriat (1967), Diewert (1973), and Varian (1982), we know that we can interpret the choices in $\mathcal{D}$ as coming from the maximization of a locally non-satiated utility if, and only if, $\mathcal{D}$ satisfies GARP. Moreover, we can choose such utility to be strictly increasing, continuous, and concave. However, GARP does not assure the possibility of thinking of the choices as coming from a utility that generates a demand function (instead of a correspondence). Matzkin and Richter (1991) show that $\mathcal{D}$ can be rationalized by a strictly concave utility if, and only if, it satisfies SARP. Strict concavity of the utility function implies that the consumer's demand is a function instead of a correspondence; this is, that for any price vector p, there is a unique optimal bundle $x^\star$.[2,3] Taking a further step, Lee and Wong

[1] We work with the following notation and terminology: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ the set of real numbers; $\mathbb{R}_+$ is the set of positive numbers including zero, and $\mathbb{R}_{++}$ excludes it. For any $M \in \mathbb{N}$, $[M]$ is the set of the first M natural numbers. A vector $x \in \mathbb{R}^M$ is $x = (x_1, x_2, \ldots, x_M)$, and $||x||$ is its Euclidean norm. The vectors $\mathbf{0}$ and $\mathbf{1}$ have all their components equal to zero and one, respectively. For any two vectors $x, y \in \mathbb{R}^M$ we write $x \geq y$ if $x_i \geq y_i$ for all $i \in [M]$, $x > y$ if $x \geq y$ and $x \neq y$, and $x \gg y$ if $x_i > y_i$ for all $i \in [M]$ ($<$, $\leq$, and $\ll$ are defined similarly). A function $f : \mathbb{R}^M \to \mathbb{R}$ is strictly increasing [strictly decreasing] if $x > y$ implies $g(x) > [<]\, g(y)$.

[2] To see this, denote the utility by U and the optimal choice by $x^\star$. By contrapositive take $x \neq x^\star$ satisfying $p\, x \leq 1$ and $U(x) = U(x^\star)$. Let $\alpha \in (0, 1)$ and $\hat{x} = \alpha x^\star + (1 - \alpha)x$. Then $p\, \hat{x} \leq 1$, and by strict concavity $U(\hat{x}) > U(x^\star)$. Therefore $x^\star$ is not optimal.

[3] Although some utilities are not strictly concave and generate an infinitely differentiable demand (like

(2005) shows that SARP is also necessary and sufficient for the existence of a utility that generates an infinitely differentiable demand.[4] A differentiable demand function is a widely used assumption in economics.

If $\mathcal{D}$ fails GARP, no (meaningful) utility function is consistent with the choices.[5] In this case, Halevy et al. (2018) propose to recover a utility function using partial efficiency, a concept proposed by Afriat (1973) and extended by Varian (1990). Partial efficiency requires each choice x^i to be preferred to bundles whose cost at prices p^i is only a share $v_i \in [0,1]$ of the income. The collection of all such shares is the N-dimension vector $\mathbf{v} = (v_1, \ldots, v_N)$, and the revealed preferences are defined accordingly.

Definition 1. Take $\mathbf{v} \in [0,1]^N$, a choice x^i, and a bundle $x \in \mathbb{R}^K_+$. x^i is

- $\mathbf{v}$-*directly revealed preferred* to x, denoted $x^i \succsim^D_{\mathbf{v}} x$, if $x^i = x$ or $p^i x \leq v_i$;
- $\mathbf{v}$-*directly revealed strictly preferred* to x, denoted $x^i \succ^D_{\mathbf{v}} x$, if $p^i x < v_i$;
- $\mathbf{v}$-*revealed preferred* to x, denoted $x^i \succsim_{\mathbf{v}} x$, if there exists a sequence of choices $(x^{k_\ell})_{\ell=1}^L$, $k_\ell \in [N]$, such that $x^i \succsim^D_{\mathbf{v}} x^{k_1} \succsim^D_{\mathbf{v}} x^{k_2} \succsim^D_{\mathbf{v}} \ldots \succsim^D_{\mathbf{v}} x^{k_L} \succsim^D_{\mathbf{v}} x$; and
- $\mathbf{v}$-*revealed strictly preferred* to x, denoted $x^i \succ_{\mathbf{v}} x$, if there exist choices x^m, $x^{m'}$ such that $x^i \succsim_{\mathbf{v}} x^m \succ^D_{\mathbf{v}} x^{m'} \succsim_{\mathbf{v}} x$.

We write $x^i \not\succsim^D x^j$ to denote that x^i is not directly revealed preferred to x^j and use a similar notation for the other revealed preferences.

The revealed preference relations in Definition 1 compare each choice x^i only with bundles affordable at prices p^i and income $v_i \in [0,1]$, instead of the original income of 1. As v_i decreases, the bundles that we compare x^i with shrink, decreasing the possibility of interpreting x^i as preferred to another bundle. If $\mathbf{v} = \mathbf{1}$, Definition 1 is equivalent to the classical definition of revealed preferences. As with the classical definition of GARP, we are interested in whether the data we observe can be thought of as coming from a (meaningful) utility.

the Leontieff utility), such cases cannot be identified under linear prices.

[4] If $\mathcal{D}$ fails SARP, the demand is not a function but a correspondence. Thus the classical idea of differentiability does not apply. Although concepts analogous to differentiability have been proposed for correspondences (e.g., Khastan et al., 2021), they are beyond the scope of this paper.

[5] A constant utility always rationalizes $\mathcal{D}$.

Definition 2. $\mathcal{D}$ is $\mathbf{v}$-*rationalizable* by the utility $U : \mathbb{R}^K_+ \to \mathbb{R}$ if $U(x^i) \geq U(x)$ whenever $p^i\, x \leq v_i$; such utility $\mathbf{v}$-*rationalizes* $\mathcal{D}$. If $U(x^i) > U(x)$ whenever $p^i \cdot x \leq 1$ and $x \neq x^i$, we say that U *strongly* $\mathbf{v}$-*rationalizes* $\mathcal{D}$ (and $\mathcal{D}$ is *strongly* $\mathbf{v}$-*rationalizable* by U).

The idea of $\mathbf{v}$-revealed preferences leads to the following definition of data consistency.

Definition 3. Take $\mathbf{v} \in [0, 1]^N$. $\mathcal{D}$ satisfies the *Generalized Axiom of Revealed Preferences given* $\mathbf{v}$ (GARP$_\mathbf{v}$) if for every pair of choices x^i, x^j

$$x^i \succsim_\mathbf{v} x^j \implies x^j \not\succ^D_\mathbf{v} x^i\,.$$

If $\mathbf{v} = \mathbf{1}$, Definition 2 and Definition 3 are equivalent to the classical definitions of rationalization and GARP, respectively. Hence, we refer to $\mathbf{1}$-rationalization as rationalization GARP$_\mathbf{1}$ simply as GARP.

From Halevy et al. (2018), we know that Afriat's (1967) theorem can be extended to partial efficiency according to $\mathbf{v}$; this is, $\mathcal{D}$ satisfies GARP$_\mathbf{v}$ if and only if it is $\mathbf{v}$-rationalized by a strictly increasing, continuous, and concave utility. The following section explores when such a utility can generate an infinitely differentiable demand function.

2.2 Partial Efficiency SARP

In the same spirit of Definition 3, we propose a partial efficiency version of SARP.

Definition 4. Take $\mathbf{v} \in [0, 1]^N$. $\mathcal{D}$ satisfies the *Strong Axiom of Revealed Preferences given* $\mathbf{v}$ (SARP$_\mathbf{v}$) if for every two choices x^i, x^j, whenever $x^i \neq x^j$

$$x^i \succsim_\mathbf{v} x^j \implies x^j \not\succsim^D_\mathbf{v} x^i$$

Again, SARP$_\mathbf{1}$ is equivalent to Houthakker's (1950) axiom, and hence we refer to it as SARP. The following remark shows that although SARP$_\mathbf{v}$ only compares different bundles, it does not present inconsistencies regarding two observations with the same choice.

Remark 1. If $\mathcal{D}$ satisfies SARP$_\mathbf{v}$ and $x^i = x^j$, then $x^i \not\succ_\mathbf{v} x^j$.

The proofs of the remarks are in Appendix A. A smaller vector $\mathbf{v}$ implies that we interpret each choice as preferred only to cheaper bundles, which reduces the set of revealed preferences. Consequently, the requirements for $\text{SARP}_{\mathbf{v}}$ are relaxed as $\mathbf{v}$ decreases. In the limit case $\mathbf{v} = \mathbf{0}$, the requirements disappear.

Remark 2. Let $\mathbf{v}' \leq \mathbf{v}$. If $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{v}}$ then it satisfies $\text{SARP}_{\mathbf{v}'}$.

Remark 3. $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{0}}$.

Surprisingly, the equivalence between SARP and rationalization by a strictly concave utility does not hold under partial efficiency. Specifically, a strictly concave utility could $\mathbf{v}$-rationalize a data set that fails $\text{SARP}_{\mathbf{v}}$. This is shown in Example 1 and Figure 2. Furthermore, the utility presented in this example generates an infinitely differentiable demand.

Example 1. Suppose $K = N = 2$, $p^1 = (1/2, 1/4)$, $x^1 = (9/5, 2/5)$, $p^2 = (1/4, 1/2)$, and $x^2 = (2/5, 9/5)$. Take $\mathbf{v} = (13/20, 13/20)$, and $U(x) = \sqrt{(1+x_1)(1+x_2)}$. As $x^1 \succsim_{\mathbf{v}} x^2 \succsim_{\mathbf{v}}^D x^1$, $\mathcal{D}$ fails SARP. However, U is strictly concave and strongly $\mathbf{v}$-rationalizes $\mathcal{D}$.

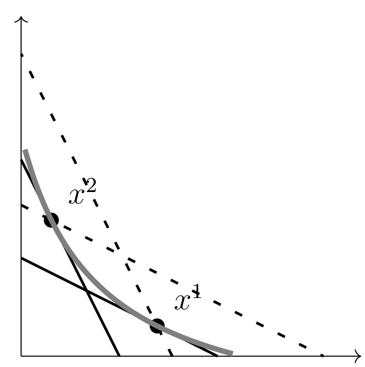


Figure 2: Example 1. $\mathcal{D}$ fails $\text{SARP}_{\mathbf{v}}$ but is $\mathbf{v}$-rationalized by a strictly concave utility.

The intuition for why $\text{SARP}_{\mathbf{v}}$ is not necessary for the existence of a demand function can be better understood starting with why failing SARP implies that there is no strictly concave utility rationalizing $\mathcal{D}$. If $\mathcal{D}$ fails SARP but satisfies GARP there are i, j such that $x^i \neq x^j$, $x^i \succsim_{\mathbf{1}} x^j$, $x^j \succsim_{\mathbf{1}}^D x^i$, and $x^j \not\succ_{\mathbf{1}}^D x^i$. This implies $p^j x^i = 1$. According to the revealed preference relation, the decision maker is indifferent between x^i and x^j. But as $p^j x^j = p^j x^i = 1$, then for $\alpha \in (0,1)$ the bundle $x^\star = \alpha x^i + (1-\alpha) x^j$ satisfies $p^j x^\star = 1$. To rationalize the data by a strictly concave U is impossible as it implies $U(x^\star) > U(x^j)$. Hence U cannot be strictly concave, and the demand has to be a correspondence. Instead, when $\mathbf{v} < 1$, $\text{SARP}_{\mathbf{v}}$ fails, and $\text{GARP}_{\mathbf{v}}$ holds, we have $x^i \succsim_{\mathbf{v}} x^j$ and $p^j x^i = v_j$. If $v_j < 1$, then $x^\star$

does not satisfy $p^j x^\star \leq v_j$ for any $\alpha \in (0,1)$; this is, $x^\star$ is not affordable at prices p^j if the income share of observation j is less than one. Therefore we cannot rule out $\mathbf{v}$-rationalization by a strictly concave utility.

The following result shows that, although not necessary, $\text{SARP}_\mathbf{v}$ is a sufficient condition for $\mathbf{v}$-rationalization by a strictly concave utility. As in the classic case, rationalization is strong.

Theorem 1. *If $\mathcal{D}$ satisfies $\text{SARP}_\mathbf{v}$, then it is strongly $\mathbf{v}$-rationalizable by a continuous, strictly increasing, and strictly concave utility.*

The proof of this result is in Appendix B. It follows the original proof of Matzkin and Richter (1991). As in their paper, it starts by showing the existence of a modified version of Afriat numbers (Lemma 2).[6] Then we use these numbers to construct a continuous, strictly increasing, and strictly concave utility that strongly $\mathbf{v}$-rationalizes the data.

Although the utility function constructed in the proof of Theorem 1 generates a demand function, such a demand is not necessarily differentiable. The following result extends the result in Lee and Wong (2005) and shows that $\text{SARP}_\mathbf{v}$ assures the existence o a utility function that generates an infinitely differentiable demand.

Proposition 1. *The utility function in Theorem 1 can be chosen to generate an infinitely differentiable demand.*

The proof of the previous result is in Appendix C. It starts from the utility function in Theorem 1 and then constructs an auxiliary data set which is ($\mathbf{1}$-)rationalized by the same utility. Then, following the proof in Lee and Wong (2005), it generates a second utility that shares the properties of U but generates an infinitely differentiable demand. Finally, from the equivalence on how both utilities compare choices in the auxiliary data set, we can conclude that the second utility strongly $\mathbf{v}$-ratinalizes $\mathcal{D}$. In the Online Appendix, we show that if we impose partial efficiency in all observations ($\mathbf{v} \ll \mathbf{1}$), we can further strengthen the result by generating an infinitely differentiable utility.

[6] We show that there are numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ such that $u^i = u^j$ whenever $x^i = x^j$ and $u^i > u^j + \lambda^i(v_i - p^i x^j)$ whenever $x^i \neq x^j$.

The following section focuses on choosing a criterion to pick the level of partial efficiency, i.e., how to select the vector $\mathbf{v}$, and whether we can distinguish between $\text{GARP}_{\mathbf{v}}$ and $\text{SARP}_{\mathbf{v}}$ with such criterion.

3 Testing Differentiable Demands

3.1 Choosing a partial efficiency level

When $\mathcal{D}$ fails GARP, we can think of the decision maker as choosing according to a meaningful utility only if we allow for partial efficiency. However, for any data set, there is a continuum of vectors $\mathbf{v}$ for which it satisfies $\text{GARP}_{\mathbf{v}}$, and since there is not a clear order between vectors in $[0,1]^N$, we need a criterion to choose a specific $\mathbf{v}$. Varian (1990) proposes to use a vector $\mathbf{v}$ that is as close as possible to $\mathbf{1}$ in some norm, using the quadratic norm as an example. Halevy et al. (2018) formalizes this notion using an aggregator function $f(\mathbf{v})$. The only requirements that we impose on $f(\mathbf{v})$ are to favor bigger vectors over smaller ones (to be strictly decreasing) and for its value to be similar when two vectors are close (to be continuous). We also normalize it such that $f(\mathbf{1}) = 0$ and $f(\mathbf{0}) = 1$.

Definition 5. Let $f : [0,1]^N \to [0,1]$ be a continuous and strictly decreasing function satisfying $f(\mathbf{1}) = 0$ and $f(\mathbf{0}) = 1$. The *Varian Inefficiency Index* $V(\mathcal{D})$ is

$$V(\mathcal{D}) = \inf_{\{\mathbf{v}\in[0,1]^N : \mathcal{D} \text{ satisfies GARP}_{\mathbf{v}}\}} f(\mathbf{v}) \,. \tag{1}$$

We refer to the Varian Inefficiency Index as the Varian Index and to f as the loss function.

The Varian Index is not the only possible criterion for choosing the level of partial efficiency; however, as discussed in Halevy et al. (2018), it is the most suitable for it. Both Afriat's (1973) Critical Cost Efficiency Index (CCEI) and the Houtman and Maks (1985) Index (HM Index) can be thought of as special cases of the Varian Index.[7] Moreover, both

[7] Afriat's (1973) CCEI imposes for all the components of the vector $\mathbf{v}$ to have the same value. Houtman and Maks (1985) impose that each component has to be either zero or one. The CCEI remains the most popular in the literature, mostly because the Varian Index is computationally more demanding: Smeulders et al. (2014) show that it is NP-Hard. Recently, Demuynck and Rehbeck (2021) developed mixed-integer

these indices will result in lower efficiency, i.e., a lower $\mathbf{v}$, and hence a higher loss. Another alternative is the Minimum Cost Index (MCI) (Dean & Martin, 2016). However, the MCI counts twice the loss of shrinking one budget set if shrinking it opens two violations of GARP; furthermore, when it counts each loss only once, it reduces to a particular case of the Varian Index. Finally, the Money Pump Index (Echenique et al., 2011) takes the average level of partial efficiency needed to satisfy GARP instead of the minimum level; hence it is not a criterion to choose the vector $\mathbf{v}$.

3.2 Preference Recoverability

Our main question is how to use the Varian Index to recover a utility that we can interpret as driving the choices (under partial efficiency) and whether the such utility can generate a differentiable demand. For example, such a utility can be used to understand the costs of parametric assumptions (Halevy et al., 2018; Zrill, 2021), to measure welfare, and to obtain information for normative criteria in individual decision-making (Kariv & Silverman, 2013).

We start by analyzing the additional loss of imposing SARP under partial efficiency. We find that there is no loss at all. If we modify the Varian Index and ask $\mathcal{D}$ to satisfy $\text{SARP}_{\mathbf{v}}$ instead of $\text{GARP}_{\mathbf{v}}$, it does not change the value of the index.

Proposition 2.

$$V(\mathcal{D}) = \inf_{\{\mathbf{v}\in[0,1]^N : \mathcal{D} \text{ satisfies } \text{SARP}_{\mathbf{v}}\}} f(\mathbf{v})\,.$$

Given the definition of the Varian Index and Proposition 2, the natural approach to recover preferences would be first to find $\mathbf{v}$ satisfying $f(\mathbf{v}) = V(\mathcal{D})$ (which exists by the intermediate value theorem) and then to analyze the utilities that $\mathbf{v}$-rationalize $\mathcal{D}$. However, as the Varian Index is an infimum, it might be the case that there is no $\mathbf{v}$ for which $V(\mathcal{D}) = f(\mathbf{v})$ and $\mathcal{D}$ satisfies $\text{GARP}_{\mathbf{v}}$. The following result shows that the latter is the case.

Proposition 3. *If $\mathcal{D}$ fails* GARP, *then for any $\mathbf{v}$ satisfying $f(\mathbf{v}) = V(\mathcal{D})$ it also fails* $\text{GARP}_{\mathbf{v}}$ *(and hence* $\text{SARP}_{\mathbf{v}}$*).*

linear programming methods to compute the Varian Index and the HM Index and showed that the indices can be quickly computed for datasets regularly collected in experiments.

Figure 1 shows a simple example that illustrates Proposition 3. In this case we have $x^1 \succ_{\mathbf{1}}^{D} x^2$ and $x^2 \succ_{\mathbf{1}}^{D} x^1$, which is a violation of SARP and GARP. Assume without loss that $f((1, p^2x^1)) < f((p^1x^2, 1))$, i.e., that it is less costly to shrink the budget set of the second observation. As for every $\varepsilon > 0$ small enough we have that $x^2 \not\succsim_{(1,p^2x^1-\varepsilon)} x^1$, $\mathcal{D}$ satisfies $\text{GARP}_{(1,p^2x^1-\varepsilon)}$. Hence $V(\mathcal{D}) = f((1, p^2x^1))$. Finally, for $\varepsilon = 0$ we have $x^1 \succ^{D}_{(1,p^2x^1)} x^2$ and $x^2 \succsim^{D}_{(1,p^2x^1)} x^1$, hence $\text{GARP}_{(1,p^2x^1)}$ fails.

Even though a partial efficiency vector $\mathbf{v}^\star$ satisfying $f(\mathbf{v}^\star) = V(\mathcal{D})$ cannot recover preferences, they can be recovered using a vector $\mathbf{v}$ that, although smaller than $\mathbf{v}^\star$, is arbitrarily close to it. Our main result, Theorem 2, (which is a direct consequence of Proposition 2 and Proposition 3) shows that the same can be done for SARP.

Theorem 2. *For every* $\mathbf{v} < \mathbf{1}$ *such that* $\mathcal{D}$ *satisfies* $\text{GARP}_{\mathbf{v}}$*, there is* $\mathbf{v}^\star$ *such that* $f(\mathbf{v}^\star) = f(\mathbf{v})$ *and* $\mathcal{D}$ *satisfies* $\text{SARP}_{\mathbf{v}^\star}$*.*

Theorem 2 implies that, if $\mathcal{D}$ fails GARP, then for every vector $\mathbf{v}$ for which it is $\mathbf{v}$-rationalizable there is another vector $\mathbf{v}^\star$ that yields the same cost as $\mathbf{v}$, and for which we can find a utility function that $\mathbf{v}^\star$ rationalizes $\mathcal{D}$ and generates an infinitely differentiable demand. This result fully characterizes the test to falsify the existence of a demand function instead of a correspondence, and the differentiability of such demand, using partial efficiency. Specifically, it implies that whenever $\mathcal{D}$ fails GARP, it is impossible to test for these characteristics. Thus, such a demand can be falsified only in a particular case: $\mathcal{D}$ has to satisfy GARP but not SARP.

3.3 Empirical Implementation

The final question we address is how usual it is to be able to falsify a differentiable demand. Theoretically, the answer to this question will depend on the data-generating process (DGP) of the price vectors that generate the budget sets and the DGPs generating the choice in each budget set. For example, for any data set in which the budget sets are all different, and the choice in each budget set is a continuous random variable, we know that to have two different observations i, j such that $p^i x^j = 1$ is a zero probability event. Hence (almost surely), any data set satisfying GARP will also satisfy SARP, so convexity cannot be tested.

We empirically analyze the possibility of falsifying the existence of a differentiable demand function using experimental data from 322 subjects, coming from the experiments in Ahn et al. (2014) and Dembo et al. (2021). Each subject makes 50 different choices under the design of Choi et al. (2007): they face a budget set to choose Arrow securities for three states of the world. We study choices with three states ($K = 3$) because it is impossible to identify GARP from SARP if there are only two states and all prices differ.[8,9] In each choice, the computer randomly selects a budget set satisfying that all components of the price vector are greater than $1/100$ (all intercepts lie between 0 and 100). At least one of them is less than $1/50$ (one intercept is greater than 50). Of the total sample, the 168 subjects from Dembo et al. (2021) knew that all the states had equal probability. The 154 subjects from Ahn et al. (2014) knew that one state had probability $1/3$ but did not know the probabilities of the other two (besides the fact that they added to $2/3$). At the end of the experiment, the computer randomly chose one choice and one state of the world, and the subject received payment according to the securities she bought.

The main finding of our analysis is that no subject satisfies GARP and fails SARP. Hence we cannot rule out the existence of a demand function, neither an infinitely differentiable one, for any of them. The specificity of the case in which these properties can be tested, along with the fact that we do not observe it in the data, suggest that (under partial efficiency)moving from a demand correspondence to an infinitely differentiable demand function does not carry a cost. We interpret this as a strong signal that assuming infinitely differentiable demand functions should not be a concern in applied economic research.

4 Final Remarks

One of the most widespread assumptions in economic research, both theoretical and empirical, is that consumers' preferences can be described by a utility function that generates a single-valued demand function instead of a correspondence. Furthermore, such demand is usually assumed to have some degree of smoothness to facilitate comparative statics. In this

[8] If $K = 2$ and two observations $i, j \in [N]$ are such that $p^i \neq p^j$ and $p^i x^j = 1$, then $x^i = x^j$.

[9] Dembo et al. (2021) discuss other advantages of experiments with three instead of two states.

paper, we study the possibility of empirically testing such assumptions, focusing on cases when the observed choices are not perfectly aligned with the agent's underlying preferences.

From Afriat (1967), Matzkin and Richter (1991), and Lee and Wong (2005), we know that for rationalizable choices, i.e., choices that satisfy GARP, SARP is the test for the existence of a demand function. If the data fails SARP, any utility rationalizing the data will generate only a demand correspondence; if the data satisfies SARP, then there is a utility that rationalizes the data and generates a demand that is a function and, moreover, infinitely differentiable. We expand this analysis by recovering preferences through partial efficiency, the most popular tool to analyze choices that fail GARP. We do so by first analyzing SARP under partial efficiency. We find that, although not necessary, partial efficiency SARP is sufficient for a differentiable demand.

Our main result shows that if choices fail GARP, it is impossible to differentiate between a demand correspondence and an infinitely differentiable demand function. Using partial efficiency, we can always choose a rationalizing utility that generates a single-valued and differentiable demand. We test the existence of such properties in experimental data and find that they cannot be falsified in any of the 322 subjects analyzed. We interpret this as evidence that the widely used assumption of an infinitely differentiable demand function demand does not carry an empirical cost.

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APPENDIX

A Remarks

Proof of Remark 1. By contrapositive suppose $x^i \succ_{\mathbf{v}} x^j$. Then there are m, m' such that $x^i \succsim_{\mathbf{v}} x^m \succ^D_{\mathbf{v}} x^{m'} \succsim_{\mathbf{v}} x^j$. As $p^m x^m = 1$ and $v_m \leq 1$, then $x^m \neq x^{m'}$. Furthermore, $x^i = x^j$

implies $x^j \succsim^D_{\mathbf{v}} x^i$, thus $x^{m'} \succsim_{\mathbf{v}} x^m$, a violation of SARP$_{\mathbf{v}}$. $\square$

Proof of Remark 2. As $\mathbf{v}' \leq \mathbf{v}$, $x^i \succsim^D_{\mathbf{v}'} x$ implies $x^i \succsim^D_{\mathbf{v}} x$. Hence $x^i \succsim_{\mathbf{v}'} x$ implies $x^i \succsim_{\mathbf{v}} x$. Suppose SARP$_{\mathbf{v}}$ holds and $x^i \succsim_{\mathbf{v}'} x^j$. Then $x^i \succsim_{\mathbf{v}} x^j$, which by SARP$_{\mathbf{v}}$ implies $x^j \not\succsim^D_{\mathbf{v}} x^i$. Hence $x^j \not\succsim^D_{\mathbf{v}'} x^i$ and SARP$_{\mathbf{v}'}$ holds. $\square$

Proof of Remark 3. $p^i x^i = 1$ implies $x^i > \mathbf{0}$; hence, as $p^j \gg \mathbf{0}$, we have $p^j x^i > 0 = v_j$. Therefore $x^i \neq x^j$ implies $x^i \not\succsim^D_{\mathbf{v}} x^j$ and SARP$_{\mathbf{0}}$ holds vacuously. $\square$

B Proof of Theorem 1

Lemma 1. *If $\mathcal{D}$ satisfies* SARP$_{\mathbf{v}}$ *there exist $i \in [N]$ such that $p^i x^j > v_i$ for all $x^j \neq x^i$.*

Proof. By counterpositive suppose that for every i there is $x^j \neq x^i$ satisfying $p^i x^j \leq v^i$. Then there is an infinite sequence $(x^{j_\ell})_{\ell=1}^\infty$ such that $x^{j_\ell} \neq x^{j_{\ell+1}}$ and $x^{j_\ell} \succsim^D_{\mathbf{v}} x^{j_{\ell+1}}$ for all ℓ. As $N < \infty$ there is $m \in [N]$ and $\ell, \ell' \in \mathbb{N}$ such that $x^m = x^{j_\ell} = x^{j_{\ell'}}$ and $\ell' \geq \ell + 2$. Thus $x^m \succsim^D_{\mathbf{v}} x^{\ell+1}$ and $x^{\ell+1} \succsim_{\mathbf{v}} x^{\ell'} = x^m$. Therefore SARP$_{\mathbf{v}}$ fails. $\square$

Lemma 2. *If $\mathcal{D}$ satisfies* SARP$_{\mathbf{v}}$ *then there exist numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ for $i \in [N]$ such that for all $i, j \in [N]$*

$$\begin{aligned} &u^i > u^j + \lambda^i (v_i - p^i x^j) && \textit{whenever } x^i \neq x^j\textit{; and} \\ &u^i = u^j && \textit{whenever } x^i = x^j. \end{aligned} \tag{2}$$

Proof. We proceed by induction on N. If $N = 1$, it is clear that $u^1 = \lambda^1 = 1$ satisfy (2).

Suppose (2) holds for all data sets with $N-1$ or less observations. Take $m \in [N]$ satisfying $p^m x^i > v_m$ whenever $x^i \neq x^m$ (which exists by Lemma 1), and define $O = [N] \setminus \{m\}$. By induction hypothesis there are numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ such that (2) holds for all $i, j \in O$. Define $B = \{i \in O : x^i \neq x^m\}$ and $E = O \setminus B$.

- If $B = \emptyset$ set $u^i = \lambda^i = 1$ for all $i \in [N]$. Clearly (2) holds for $\mathcal{D}$.
- If $B \neq \emptyset$, set $u^m = \min_{i \in B} u^i + \lambda^i (p^i x^m - v_i) - \varepsilon$ if $E = \emptyset$, and if $E \neq \emptyset$, take $\ell \in E$ and set $u^m = u^\ell$. In both cases $u^i > u^m + \lambda^i (v_i - p^i x^m)$ whenever $i \in B$, and $u^m = u^i$ whenever $i \in E$.

Set

$$\lambda^N = \max\left\{\max_{i\in B}\frac{u^i - u^N + \varepsilon}{p^N x^i - v_N}\,;\,1\right\}\,.$$

Then $\lambda^N \geq 1 > 0$ and $u^N > u^i + \lambda^N(v_N - p^N x^i)$ for all $i \in B$ (since $p^N x^i - v_N > 0$). Therefore (2) holds for $\mathcal{D}$.

□

Proof of Theorem 1. Set $M > 0$, and define $g(x) = (M + ||x||^2)^{1/2} - M^{1/2}$. As $\mathcal{D}$ satisfies $\text{SARP}_{\mathbf{v}}$ there are numbers $u^i \in \mathbb{R}$ and $\lambda^i > 0$ such that (2) holds (Lemma 2). Furthermore, as $\lambda^i p^i_k > 0$ there is $\varepsilon > 0$ such that

$$u^i - \varepsilon g(x^i - x^j) > u^j + \lambda^i(v_i - p^i x^j) \qquad \text{whenever } x^i \neq x^j\text{; and} \tag{3}$$

$$\lambda^i p^i_k > \varepsilon \qquad \text{for all } i \in [N], k \in [K]\,. \tag{4}$$

Let

$$\phi^i(x) = u^i - \lambda^i(v_i - p^i x) - \varepsilon g(x - x^i) \qquad \text{for each } i \in [N]\text{, and}$$

$$U(x) = \min_{i\in[N]} \phi^i(x)\,.$$

Since each ϕ^i is continuous, strictly concave, and strictly increasing, $U(x)$ inherits these properties.[10]

Let m be a minimizer of $U(x^i)$. If $x^i = x^m$ then $u^i = u^m$ and

$$U(x^i) = u^m - \lambda^m(v_m - p^m x^i) - \varepsilon g(x^i - x^m) = u^i - \lambda^m(v_m - p^m x^m) \geq u^i\,.$$

The second equality follows from $g(0) = 0$ and the inequality from $p^m x^m \geq v_m$ and $\lambda^m > 0$. If $x^m \neq x^i$ from (3) we have

$$U(x^i) = u^m - \lambda^m(v_m - p^m x^i) - \varepsilon g(x^i - x^m) > u^i\,.$$

Hence $U(x^i) \geq u^i$.

[10] $\phi^i(\cdot)$ is strictly increasing since, from (4), for all $k \in [K]$

$$\frac{\partial \phi^i(x)}{\partial x_k} = \lambda^i p^i_k - \varepsilon\left(\frac{(x_k)^2}{M + \sum_{k\in[K]}(x_k)^2}\right) > \lambda^i p^i_k - \varepsilon > 0\,.$$

Take $x \neq x^i$ such that $p^i x \leq v_i$. Then

$$\begin{aligned}
U(x) &= \min_{j \in [N]} u^j - \lambda^j (v_j - p^j x) - \varepsilon g(x - x^j) \\
&\leq u^i - \lambda^i (v_i - p^i x) - \varepsilon g(x - x^i) && \text{(since } i \in [N]\text{)} \\
&\leq u^i - \varepsilon g(x - x^i) && \text{(since } \lambda^i > 0 \text{ and } p^i x \leq v_i\text{)} \\
&< u^i && \text{(since } \varepsilon > 0 \text{ and } x \neq x^i\text{)} \\
&\leq U(x^i) && \text{(since } U(x^i) \geq u^i\text{)}.
\end{aligned}$$

Therefore U strongly $\mathbf{v}$-rationalizes $\mathcal{D}$. □

C Proof of Proposition 1

Lemma 3. *Let U be a continuous, strictly concave, and strictly increasing utility generating a demand $m : \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K_+$. For every $x \in \mathbb{R}^K_+ \setminus \{\mathbf{0}\}$ there is $p \gg \mathbf{0}$ such that $m(p, 1) = x$.*

Proof. Take $x \in \mathbb{R}^K_+ \setminus \{\mathbf{0}\}$. Since U is strictly concave it has a supergradient b at x; this is, $U(x) > U(y) + b(x - y)$ whenever $y \neq x$. We first show, by contradiction, that $b \gg \mathbf{0}$. Suppose $b_k \leq 0$ for some $k \in [K]$, denote by e^k the vector with k^{th} component equal to one and all the others equal to zero, and set $y = x + e^k > x$. As U is strictly increasing

$$U(y) + b(x - y) = U(y) - b\, e^k = U(y) - b_k \geq U(y) > U(x)\,,$$

a contradiction.

Define $p = (b\, x)^{-1} b \gg \mathbf{0}$. Then $p\, x = 1$. Moreover, $y \neq x$ and $p\, y \leq 1$ imply $U(x) > U(y)$ (since $b\, x \leq b\, y$). Therefore $x = m(p, 1)$. □

Proof of Proposition 1. Suppose $\mathcal{D}$ satisfies SARP$_{\mathbf{v}}$ and let U be a continuous, strictly concave, and strictly increasing utility $\mathbf{v}$-rationalizing $\mathcal{D}$ (Theorem 1). Let $m(p, e)$ be the demand function generated by U. By strong $\mathbf{v}$-rationalization, for every $i \in [N]$ we have $U(x^i) \geq U(m(p^i, v_i))$, with strict inequality if $v_i < 1$. Construct the data set $\widetilde{\mathcal{D}} = (\widetilde{p}^j, \widetilde{x}^j)_{j \in [J]}$ as follows: for every i in $[N]$

- If $v_i = 1$, add an observation $(\widetilde{p}^j, \widetilde{x}^j) = (p^i, x^j)$.

- If $v_i < 1$, add two observations $(\widetilde{p}^j, \widetilde{x}^j)$ and $(\widetilde{p}^{j'}, \widetilde{x}^{j'})$, where:
  - $\widetilde{p}^j = p^i$, $\widetilde{x}^j = m(p^i, v_i)$, and
  - $\widetilde{x}^{j'} = x^i$, $\widetilde{p}^{j'} = p$ for some p such that $m(p, 1) = x^i$, which exists by Lemma 3.

By construction $\widetilde{\mathcal{D}}$ is strongly $\mathbf{1}$-rationalized by U, hence it satisfies SARP. By Lee and Wong (2005) there is an strictly increasing, strictly concave $\widetilde{U}$ that strongly $\mathbf{1}$-rationalizes $\widetilde{\mathcal{D}}$ and generates an infinitely differentiable demand. Furthermore, from their proof we can choose $\widetilde{U}$ agreeing with U on how to compare choices in $\widetilde{\mathcal{D}}$, i.e.,

$$\widetilde{U}(x^i) \geq \widetilde{U}(x^j) \iff U(x^i) \geq U(x^j) \qquad \text{whenever } i, j \in [J]$$

Finally, take any $i \in [N]$.

- If $v_i = 1$ then there is $j \in [J]$ such that $(\widetilde{p}^j, \widetilde{x}^j) = (p^i, x^i)$. By strong rationalization of $\widetilde{\mathcal{D}}$ we have that $\widetilde{U}(x^i) > \widetilde{U}(x)$ whenever $p^i x \leq v_i$ and $x \neq x^i$.
- If $v_i < 1$ then there are $j, j' \in [J]$ such that $\widetilde{p}^j = p^i$, $\widetilde{x}^j = m(p^i, v_i)$, $\widetilde{x}^{j'} = x^i$, and $m(\widetilde{p}^{j'}, 1) = x^i$. As $v_i < 1$, strong rationalization of $\mathcal{D}$ by U implies $U(x^i) > U(m(p^i, v_i)) = U(\widetilde{x}^j)$. Moreover, as $\widetilde{U}$ and U agree on how to rank choices in $\widetilde{\mathcal{D}}$, we have $\widetilde{U}(x^i) > \widetilde{U}(\widetilde{x}^j)$. Strong rationalization of $\widetilde{\mathcal{D}}$ by $\widetilde{U}$ implies that $\widetilde{U}(x^i) > \widetilde{U}(\widetilde{x}^j) \geq \widetilde{U}(x)$ whenever $p^i x \leq v_i$.

Therefore $\widetilde{U}$ strongly $\mathbf{v}$-rationalizes $\mathcal{D}$.

□

D Proof of Proposition 2

Lemma 4. *If* $\text{GARP}_{\mathbf{v}}$ *holds then there is a sequence* $(\mathbf{v}^n)_{n \in \mathbb{N}}$ *such that*

1. $\mathbf{v}^n \leq \mathbf{v}^{n+1}$ *for all* n*;*
2. $\mathbf{v}^n \to \mathbf{v}$*; and*
3. $\mathcal{D}$ *satisfies* $\text{SARP}_{\mathbf{v}^n}$ *for all* n*.*

Proof. Suppose $\text{GARP}_{\mathbf{v}}$ holds, and let $C = \{(i, j) \in [N] \times [N] : x^i \neq x^j,\ x^i \succsim_{\mathbf{v}} x^j, \text{ and } x^j \succsim_{\mathbf{v}}^D x^i\}$. As $\mathcal{D}$ satisfies $\text{GARP}_{\mathbf{v}}$, then $(i, j) \in C$ implies $x^j \not\succ_{\mathbf{v}}^D x^i$, thus $v_j = p^j x^i > 0$. Define $\mathbf{v}^n$ by

$$v_j^n = \begin{cases} \frac{n}{n+1} v_j & \text{if } (i, j) \in C \text{ for some } i \in [N] \\ v_j & \text{otherwise.} \end{cases}$$

Then $\mathbf{v}^n \leq \mathbf{v}^{n+1}$ for all n, and $\mathbf{v}^n \to \mathbf{v}$. Moreover, if $(i,j) \in C$ then $v_j^n < v_j$.

Finally, suppose $x^i \neq x^j$ and $x^i \succsim_{\mathbf{v}^n} x^j$. If $(i,j) \notin C$ then $x^j \not\succsim_{\mathbf{v}}^D x^i$, and $p^j x^i > v_j \geq v_j^n$. If $(i,j) \in C$ then $p^j x^i = v_i > v_i^n$. Hence $x^j \not\succsim_{\mathbf{v}^n}^D x^i$, therefore $\mathcal{D}$ satisfies SARP$_{\mathbf{v}^n}$. □

Proof of Proposition 2. As SARP$_{\mathbf{v}}$ is stronger than GARP$_{\mathbf{v}}$,

$$\inf_{\{\mathbf{v} \in [0,1]^N : \mathcal{D} \text{ satisfies SARP}_{\mathbf{v}}\}} f(\mathbf{v}) \geq V(\mathcal{D}) \,. \tag{5}$$

By definition of $V(\mathcal{D})$ there is a sequence $\mathbf{v}^n \to \mathbf{v}^\star$ such that GARP$_{\mathbf{v}^n}$ holds for all n and $f(\mathbf{v}^\star) = V(\mathcal{D})$. By Lemma 4, for each n there is a sequence $(\mathbf{b}^{n,i})_{i \in \mathbb{N}}$ such that $\mathbf{b}^{n,i} \to \mathbf{v}^n$ and $\mathcal{D}$ satisfies SARP$_{\mathbf{b}^{n,i}}$ for every $i, n \in \mathbb{N}$. Set $\varepsilon > 0$ and for each n take $j(n)$ such that $|| \, \mathbf{v}^n - \mathbf{b}^{n,j(n)} || < \varepsilon/n$. Define the sequence $(\mathbf{c}^n)_{n \in \mathbb{N}}$ by $\mathbf{c}^n = \mathbf{b}^{n,j(n)}$. Then $\mathcal{D}$ satisfies SARP$_{\mathbf{c}^n}$ for all n and $\mathbf{c}^n \to \mathbf{v}^\star$. Continuity of f implies $f(\mathbf{c}^n) \to f(\mathbf{v}^\star)$, thus

$$\inf_{\{\mathbf{v} \in [0,1]^N : \mathcal{D} \text{ satisfies SARP}_{\mathbf{v}}\}} f(\mathbf{v}) \leq f(\mathbf{v}^\star) = V(\mathcal{D}) \,. \tag{6}$$

(5) and (6) imply the desired result. □

E Proposition 3

This proof uses a criteria of *almost* data consistency developed by Polisson et al. (2020, Appendix A9.1).

Definition 6. For $\mathbf{v} \in [0,1]^N$, $\mathcal{D}$ *almost satisfies* GARP$_{\mathbf{v}}$ (i.e., it satisfies aGARP$_{\mathbf{v}}$) if there is a sequence $(\mathbf{v}^n)_{n \in \mathbb{N}}$ such that

1. $\mathbf{v}^n \leq \mathbf{v}$;
2. $\mathbf{v}^n \to \mathbf{v}$; and
3. $\mathcal{D}$ satisfies GARP$_{\mathbf{v}^n}$ for all $n \in \mathbb{N}$.

Lemma 5. *$\mathcal{D}$ satisfies* aGARP$_{\mathbf{v}}$ *if, and only if, when restricted to the choices in $\mathcal{D}$, $\succ_{\mathbf{v}}^D$ is acyclic.*

Proof. See Polisson et al. (2020, Appendix A9.1). □

Lemma 6. *If $\mathcal{D}$ satisfies* aGARP$_{\mathbf{v}}$, *then $V(\mathcal{D}) \leq f(\mathbf{v})$.*

Proof. It follows from the definitions of $V(\mathcal{D})$ and aGARP$_{\mathbf{v}}$. $\square$

Proof of Proposition 3. Suppose GARP fails and take $\mathbf{v}$ such that $f(\mathbf{v}) = V(\mathcal{D})$. If $\mathcal{D}$ fails aGARP$_{\mathbf{v}}$ then it also fails GARP$_{\mathbf{v}}$. If it satisfies aGARP$_{\mathbf{v}}$ then

- If $\mathbf{v} = \mathbf{1}$ then GARP$_{\mathbf{v}}$ fails by assumption.
- If $\mathbf{v} < \mathbf{1}$ then there is i such that $v_i < 1$; define $A = \{j \in [N] : x^j \succ_{\mathbf{v}} x^i\}$. Towards a contradiction suppose GARP$_{\mathbf{v}}$ holds. Then $p^i x^j > v_i$ for all $j \in A$.[11] As A is finite there is $\varepsilon > 0$ such that $p^i x^j > v_i + \varepsilon$ for all $j \in A$. Define $\mathbf{v}' \in [0,1]^N$ by

$$v'_n = \begin{cases} v_i + \varepsilon & \text{if } n = i \\ v_n & \text{otherwise.} \end{cases}$$

When restricted to choices in $\mathcal{D}$, $\succ^D_{\mathbf{v}} = \succ^D_{\mathbf{v}'}$. By Lemma 5, aGARP$_{\mathbf{v}}$ implies that $\succ^D_{\mathbf{v}}$ restricted to choices is acyclic, hence $\succ^D_{\mathbf{v}'}$ also is and aGARP$_{\mathbf{v}'}$ holds. But $\mathbf{v}' > \mathbf{v}$ implies $f(\mathbf{v}') < f(\mathbf{v}) = V(\mathcal{D})$, which contradicts Lemma 6.

As $\mathcal{D}$ fails GARP$_{\mathbf{v}}$, it also fails SARP$_{\mathbf{v}}$. $\square$

Theorem 2

Proof. As $\mathbf{v} < 1$ and GARP$_{\mathbf{v}}$ holds, Proposition 3 implies $f(\mathbf{v}) < V(\mathcal{D})$. By Proposition 2 there is $(\mathbf{v}^n)_{n \in \mathbb{N}}$ such that SARP$_{\mathbf{v}^n}$ holds for all n, and $f(\mathbf{v}^n) \to V(\mathcal{D})$. Thus $f(\mathbf{v}^{n_0}) \geq f(\mathbf{v})$ for n_0 large enough. As f is continuous and strictly decreasing, there is $\mathbf{v}^\star \leq \mathbf{v}^{n_0}$ such that $f(\mathbf{v}^\star) = f(\mathbf{v})$. Remark 2 implies that $\mathcal{D}$ satisfies SARP$_{\mathbf{v}^\star}$. $\square$

[11] If not, then $x^i \succsim^D_{\mathbf{v}} x^j$. As $x^j \succ_{\mathbf{v}} x^i$ there are m, m' such that $x^j \succsim_{\mathbf{v}} x^m \succ^D_{\mathbf{v}} x^{m'} \succsim_{\mathbf{v}} x^i$. Then $x^{m'} \succsim_{\mathbf{v}} x^m$ and $x^m \succ^D_{\mathbf{v}} x^{m'}$, and GARP$_{\mathbf{v}}$ fails.